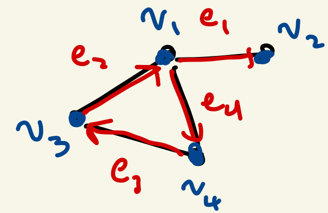
Last time ... graphs!

Given a graph with edges $e_1 \cdots e_n$

and vertices $v_1 \cdots v_m$

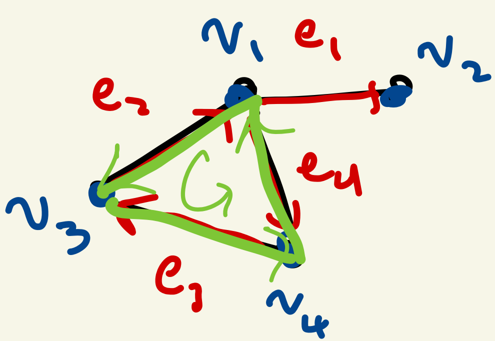


$$C_1 = \text{span}(e_1 \cdots e_n)$$

$$C_0 = \text{span}(v_1 \cdots v_n)$$

$$C_1 = \text{span}(e_1, e_2, e_3, e_4)$$

$$C_0 = \text{span}(v_1, v_2, v_3, v_4)$$



circuit $\rightsquigarrow$ $-e_2 - e_3 - e_4 \in C_1$.

$\rightsquigarrow$ $-2e_2 - 2e_3 - 2e_4$

going around the circuit 2 times

Thm Let G be a connected graph.

all vertices are connected by paths

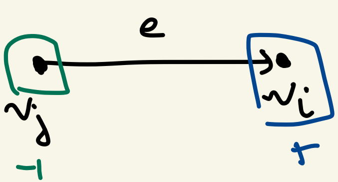
Then $\#v - \#e = 1 - \underbrace{\#\text{ ind circuits}}_{\text{what?}}$.

We have a linear function $\partial : C_1 \longrightarrow C_0$.

domain "inputs"

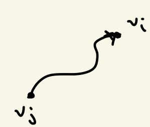
codomain "possible outputs"

Given an edge e which starts at v_j and ends at v_i



Then $\partial(e) = +v_i - v_j$

(Remember from multi:



$$\int \nabla\phi \cdot ds = \underset{+}{\phi(v_i)} - \underset{-}{\phi(v_j)}$$

)

∂ is called the <u>boundary</u> operator. $e_1 \ldots e_n$ forms a basis of C_1, so if we define ∂ on the basis it extends to a function on all linear combination

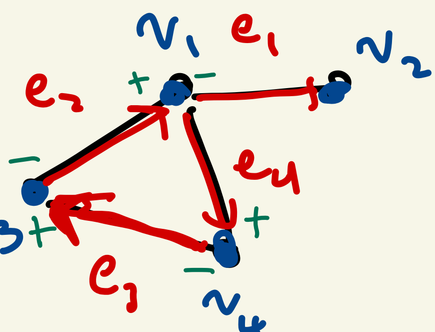
$$c_1 e_1 + \ldots + c_n e_n.$$

$$S(v_i) = T(v_i)$$

$$\Rightarrow S = T$$

$$\partial(c_1 e_1 + \ldots + c_n e_n) \overset{\text{def}}{=} c_1 \partial(e_1) + \ldots + c_n \partial(e_n)$$

<u>Ex</u>



$$\partial(-e_2 - e_3 - e_4)$$

$$= -\partial(e_2) - \partial(e_3) - \partial(e_4)$$

$$= -(v_1 - v_3) - (v_3 - v_4) - (v_4 - v_1)$$

$$= -\cancel{v_1} + \cancel{v_3} - \cancel{v_3} + \cancel{v_4} - \cancel{v_4} + \cancel{v_1} = 0!$$

General idea.

$\partial(\text{circuit}) = 0$ always!

Prop Given a circuit $c = \sum \pm e_i$

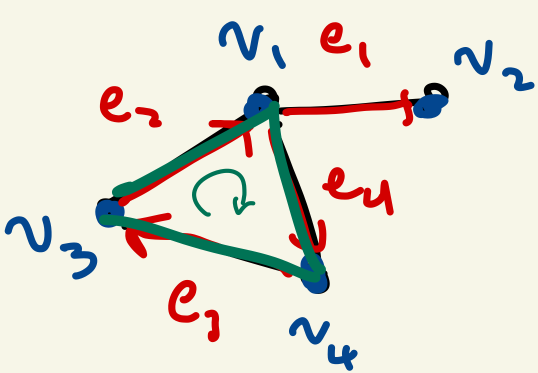
$\partial(c) = 0$.

Furthermore if $\partial(\sum c_i e_i) = 0$

then $\sum c_i e_i$ represents a circuit.

Proof by example.

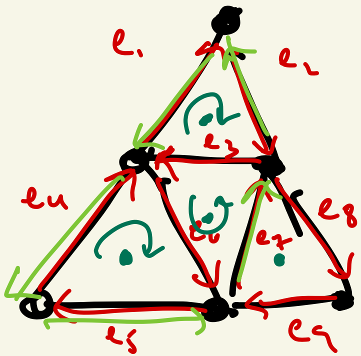
By independent circuits, I mean circuits that generate all linear combination $\sum c_i e_i$ s.t. $\partial(\sum c_i e_i) = 0$.



Here we have 1 independent circuit

$e_2 + e_3 + e_4$ generates all other circuits!

Independent circuits are the "holes" in your graph.



This graph has 4 independent circuits

$e_1 + e_2 + e_3$

$e_4 + e_5 + e_6$

$e_3 + e_6 + e_7$

$e_7 + e_8 + e_9$

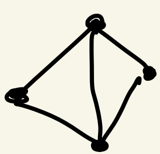
$$-e_1 - e_4 - e_5 + e_7 - e_2 = -(e_1+e_2+e_3) - (e_4+e_5+e_6) + (e_3+e_6+e_7)$$

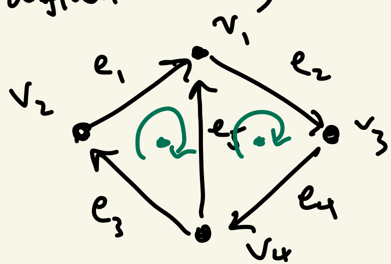
check at home!

Thm Given a connected graph

$$\#v - \#e = 1 - \underbrace{\#\text{ind circuits}}_{\#\text{ of "holes" in your graph}}$$

Pf (2.6.4b) (w/ different arrows)





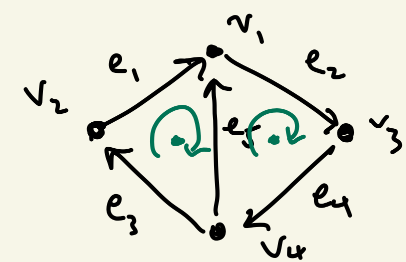
$\#v - \#e = 1 - \#$ ind circuits

$4 - 5 = -1 = 1 - \#$ ind. circ.

$\implies \#$ ind circ $= 2 = 2$ "holes"

The ind circ. $e_1 - e_5 + e_3$ and $e_2 + e_4 + e_5$.

How do you actually prove these 2 ind circuits?



$\partial : C_1 \longrightarrow C_0$

$e_1 \longmapsto v_1 - v_2$
$e_2 \longmapsto v_3 - v_1$
$e_3 \longmapsto v_2 - v_4$
$e_4 \longmapsto v_4 - v_3$
$e_5 \longmapsto v_1 - v_4$

$\partial(\text{cir}) = 0.$

$\ker \partial = 0.$

If ∂ were a matrix, we'd know how to do this!

$$\partial \longrightarrow M$$

$$\begin{array}{c|ccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 \\
\hline
v_1 & 1 & -1 & 0 & 0 & 1 \\
v_2 & -1 & 0 & 1 & 0 & 0 \\
v_3 & 0 & 1 & 0 & -1 & 0 \\
v_4 & 0 & 0 & -1 & 1 & -1
\end{array}$$

$$\partial(e_1) = v_1 - v_2$$

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

4×5 matrix

$= -A^T$

$$\boxed{\text{Incidence matrix} \quad A = -M^T}$$

$$\partial(\text{circuit}) = 0 \rightsquigarrow \underline{\ker(M)} = ??$$

$(\underline{\text{coker}(A)}$ book's notation$)$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix} \xrightarrow[\text{please use a computer}]{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & & & \uparrow & \uparrow \\ & & & \text{free} & \text{free} \end{array}$$

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$\dim(\ker(M)) = \#$ free columns $= 2$

$\ker(M) =$ all circuits $\qquad \dim(\ker(M)) = \#$ ind circuits

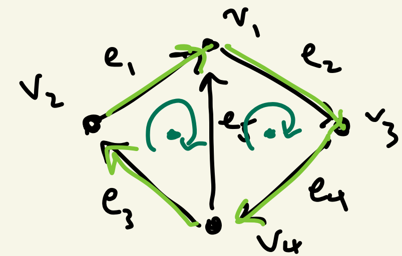
$\#$ ind circuits $= \dim(\ker(M)) = 2 = 2$ free columns

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array} \rightarrow \begin{aligned} x_1 &= x_4 - x_5 \\ x_2 &= x_4 \\ x_3 &= x_4 - x_5 \end{aligned}$$

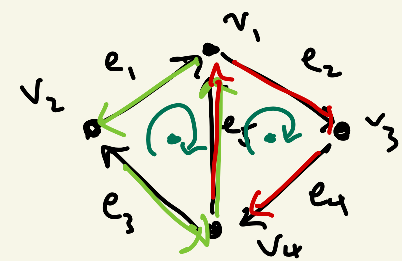
$$\longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_4 - x_5 \\ x_4 \\ x_4 - x_5 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} x_5$$

circuits! circuits!

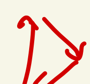
$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \rightsquigarrow e_1 + e_2 + e_3 + e_4 + \cancel{0 e_5}$$



$$\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow -e_1 - e_3 + e_5$$



$$(e_1 + e_2 + e_3 + e_4) + (-e_1 - e_3 + e_5)$$
$$= e_2 + e_4 + e_5$$



2 # ind circuits!

2.6.4 recreate this process!!

2.6.8 short proof only use # v - # e = 1 - # ind. circuits

2.6.10 also just use # v - # e = 1 - # ind circuits

(+ edge counting)

Pf rank-nullity

OH tomorrow!

7.1.19 e

$L(f) = x^2 f(x)$

what is the codomain?

what kind of object is the output?

$L: C^1[a,b] \longrightarrow \boxed{C^1[a,b]}$ 0, 1?

codomain!

$\frac{d}{dx}\left(x^2 f(x)\right) = 2x f + x^2 f'$

- $L(f+g) = L(f) + L(g)$
- $L(cf) = c\,L(f)$

$$L(f+g) = x^2 (f+g)(x) = x^2 (f(x) + g(x))$$
$$= x^2 f(x) + x^2 g(x) = L(f) + L(g)$$

$$L(cf) = x^2 (cf)(x) = x^2 c f(x) = c(x^2 f(x))$$
$$= c\,L(f)$$

It's linear!