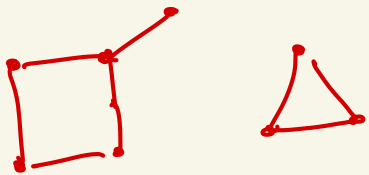
Interesting note! $\underline{\#v - \#e}$ = 1 − # ind circ "=" $\underbrace{\text{1 − \# holes in your graph}}$

formula only depends the shape of the graph



#v − #e is a "topological invariant".

$\chi(G) = \#v - \#e$ Euler characteristic

HW 9 due tonight!

Chp 8. Eigenvalues, Eigenvectors, etc.

Def: Let A be an $n \times n$ matrix. We say λ is an eigenvalue of A if there exists a nonzero $v \neq 0$ such that $Av = \lambda v$. v is called an eigenvector for λ.

Def: Let A be an $n \times n$ matrix. Let $V_\lambda = \{ v \in \mathbb{R}^n \mid Av = \lambda v \}$. We say λ is an eigenvalue of A if $V_\lambda \neq 0$. If $V_\lambda \neq 0$, then we say V_λ is a eigenspace of A. The vectors in V_λ are the eigenvectors.

$0 \in V_\lambda$.

$A \cdot \vec{0} = \lambda \cdot \vec{0}$

no matter the λ.

Prop For all λ, V_λ is a subspace of $\mathbb{R}^n$. (If λ is not an eigenval $V_\lambda = \{0\}$)

Pf Suppose $v, w \in V_\lambda$. i.e. $Av = \lambda v$
$Aw = \lambda w$

then $\vec{v}+\vec{w} \in V_\lambda$ since $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} = \lambda\vec{v} + \lambda\vec{w}$
$= \lambda(v+w)$.

WTS $c\vec{v} \in V_\lambda$ $A(c\vec{v}) = cA\vec{v} = c(\lambda\vec{v}) = \lambda(c\vec{v})$

$\vec{0} \in V_\lambda$ $A(\vec{0}) = \lambda\vec{0} = \vec{0}$

Moral of the story: V_λ is a subspace.

Def: Since V_λ is a subspace, $\dim(V_\lambda)$ is well defined.

$\dim(V_\lambda) = \#$ basis eigenvectors of V_λ = maximum # of independent eigenvectors for λ.

Given an eigenvalue λ, $\dim(V_\lambda)$ is called <u>geometric multiplicity</u> of λ.

Prop λ is an eigenvalue for A iff $\det(A - \lambda I) = 0$.

$V_\lambda = \ker(A - \lambda I)$ = set of all eigenvectors for λ.

Pf: * λ is a eigenvalue for A

$\iff Av = \lambda v$ for $v \neq 0$.

$\iff Av - \lambda v = 0 \quad v \neq 0$

$\iff Av - \lambda I v = 0 \quad v \neq 0$

$\iff (A - \lambda I)v = 0 \quad v \neq 0$

$\iff \boxed{\ker(A - \lambda I)} \neq 0$

$\iff (A - \lambda I)^{-1}$ doesn't exist $\iff \det(A - \lambda I) = 0$. *

$V_\lambda = \ker(A - \lambda I)$

set of eigenvectors

$\ker(M) = \{Mv = 0\}$

$M = A - \lambda I$

~~$(A - \lambda) v = 0$~~

matrix scalar

$\square$

Def: We call $\det(A - \lambda I)$ the characteristic polynomial of A,

it's a polynomial in the variable λ, whose solutions are the

eigenvalues.

Def: Given the $\det(A - \lambda I)$, a solution λ_i might

repeat k_i times. We call the # of times

λ_i appears as a root of $\det(A - \lambda I)$ the

algebraic multiplicity of λ_i.

$\dim(V_\lambda) \leq$ # of repeats of λ in $\det(A - \lambda I)$

— geometric mult $<$ — alg mult

is possible

Ex $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ What are the eigenvalues? ~~What are the eigenvectors?~~

What are the eigenspaces V_λ?

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1-\lambda & 3 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 6 = 0$$

$$\lambda^2 - \cancel{\lambda} + \cancel{\lambda} - 1 - 6 = 0$$

$$\lambda^2 - 7 = 0$$

$$\Longrightarrow \lambda = \pm\sqrt{7} = \sqrt{7}, -\sqrt{7}$$

We have 2 eigenvalues, $\sqrt{7}, -\sqrt{7}$, they each repeat only once as roots

alg mult = 1.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

$V_{\sqrt{7}}$ = set of all eigenvectors for $\lambda = \sqrt{7}$

$$= \ker(A - \sqrt{7}I) = \ker \begin{pmatrix} 1-\sqrt{7} & 3 \\ 2 & -1-\sqrt{7} \end{pmatrix}$$

* In the past $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{7}\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1-\sqrt{7} & 3 \\ 2 & -1-\sqrt{7} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$$x - \tfrac{1}{2}(1+\sqrt{7})y = 0$$

$$\begin{pmatrix} 1-\sqrt{7} & 3 \\ 2 & -1-\sqrt{7} \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -\frac{1}{2}(1+\sqrt{7}) \\ 0 & 0 \end{pmatrix}$$

$x = \frac{1}{2}(1+\sqrt{7})y$ $\qquad$ y free variable!

$$V_{\sqrt{7}} = \ker\begin{pmatrix} 1-\sqrt{7} & 3 \\ 2 & -1-\sqrt{7} \end{pmatrix} = \text{span}\begin{pmatrix} \frac{1}{2}(1+\sqrt{7}) \\ 1 \end{pmatrix} = \text{span}\underbrace{\begin{pmatrix} 1+\sqrt{7} \\ 2 \end{pmatrix}}_{1 \text{ vector}}$$

geom mult of $\lambda = \sqrt{7}$ = $\dim(V_{\sqrt{7}}) = 1$ = alg mult

$$V_{-\sqrt{7}} = \ker\begin{pmatrix} 1+\sqrt{7} & 3 \\ 2 & -1+\sqrt{7} \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -\frac{1}{2}(1-\sqrt{7}) \\ 0 & 0 \end{pmatrix}$$

$$V_{-\sqrt{7}} = \operatorname{span}\begin{pmatrix} \frac{1}{2}(1-\sqrt{7}) \\ 1 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1-\sqrt{7} \\ 2 \end{pmatrix}$$

$$(\text{geom mult of } \lambda = -\sqrt{7}) = 1$$

1 basis vector.

Ex $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\det(I - \lambda I) = \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 = 0 \qquad \lambda = \underbrace{1, 1}_{\text{2 times}}$$

$\lambda = 1$ is an eigenvalue but alg mult $\lambda = 2$

$$V_1 = \ker(I - 1I) = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2 = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

geom mult $= 2$

Recall <u>diagonalization</u> :

If A has a basis of eigenvectors $\vec{v}_1 \dots \vec{v}_n$.

The change of basis $\vec{e}_1 \dots \vec{e}_n \longrightarrow \vec{v}_1 \dots \vec{v}_n$ is diagonalization.

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}^{-1} A \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.$$

$\sqrt{7}$ alg = 1 = geom same for $-\sqrt{7}$

$$\begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{7} & 1-\sqrt{7} \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1+\sqrt{7} & 1-\sqrt{7} \\ 2 & 2 \end{pmatrix}$$

<u>Thm</u> : Let A be an $n \times n$ matrix. $\boxed{A \text{ is diagonalizable}}$

$\iff \mathbb{R}^n$ has a basis of eigenvectors of A $\overset{*}{\iff}$ $\boxed{\text{For all } \lambda_i \text{ geom mult} = \text{alg mult.}}$

let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = 0$

$\det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = (-\lambda)^3 = 0$

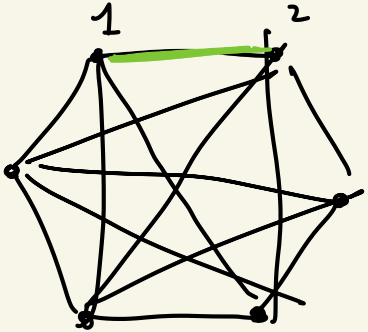
$\lambda = 0, 0, 0$ $\quad$ (alg mult of $\lambda = 0$) $= 3$

$V_0 = \ker(A - 0I) = \ker\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{1 vector}}$

$\dim(V_0) = $ (geom mult of $\lambda=0$) $= 1$

$1 \neq 3$ $\quad$ A is not diagonalizable!

3 basis vector slot but we only have 1 eigenvector to give, so no basis!



$\#$ of edges = $\#$ of pairs of distinct vertices

$$= 6 \cdot (6-1) \cdot \frac{1}{2}$$

6 choices for first one

5 choices for 2nd vertex

vertex 1 , vertex 2

vertex 2 , vertex 1

but same edge

$\#$ of edges of complete graph $= \dfrac{n(n-1)}{2} = \dbinom{n}{2}$

$L: C^1[a,b] \longrightarrow \mathbb{R}$

codomain!

7.1.19c

$L(f) = f'(1)$

sum rule!

$L(f+g) = (f+g)'(1) = f'(1) + g'(1) = L(f) + L(g)$

7.1.19

- $L(f+g) = L(f) + L(g)$

$L(cf) = cL(g)$

$L(f) = x^2 f(x)$

$L(f+g) = x^2 (f+g)(x) = x^2 (f(x) + g(x))$

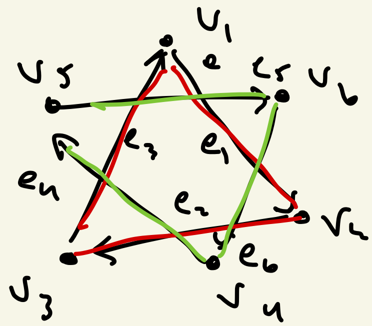
$L(f) = f'(1)$

$L(f+g) = (f+g)(x) + 2$

$L(x^2) = \frac{d}{dx}(x^2)\Big|_{x=1}$

$= 2x\Big|_{x=1} = \boxed{2}$

real number!



$$\partial(e_1) = v_2 - v_1$$

end − beginning

$$\partial(e_2) = v_3 - v_2$$

$$\partial(e_3) = v_1 - v_3$$

$$\partial(e_4) = v_5 - v_4$$

$$\partial(e_5) = v_6 - v_5$$

$$\partial(e_6) = v_4 - v_6$$

This graph is <u>not</u> connected!

$$\begin{array}{cccccc|c}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & \\
-1 & 0 & 1 & 0 & 0 & 0 & v_1 \\
1 & -1 & 0 & 0 & 0 & 0 & v_2 \\
0 & 1 & -1 & 0 & 0 & 0 & v_3 \\
0 & 0 & 0 & -1 & 0 & 1 & v_4 \\
0 & 0 & 0 & 1 & -1 & 0 & v_5 \\
0 & 0 & 0 & 0 & 1 & -1 & v_6
\end{array}$$