Final Exam: Dec 21st monday 1:30 - 3:30 pm

NEW! $V_\lambda = \{ v \in \mathbb{R}^n \mid Av = \lambda v \}$ where A is some $n \times n$ matrix and $\lambda \in \mathbb{R}$. So if $V_\lambda \neq \{0\}$, then λ is an eigenvalue and V_λ is called the eigenspace.

Thm A matrix A is diagonalizable $(D = S^{-1} A S)$
$S = (v_1 \dots v_n)$ matrix of eigenvector basis

iff

# of independent eigenvectors $=$ $\dim(V_\lambda)$ = geom mult = alg mult = # of repeats of λ in $\det(A - \lambda I)$

for all eigenvalues λ

How come A acts diagonally on a basis of eigenvectors?

Suppose $\vec{v}_1 \dots \vec{v}_n$ is a basis of eigenvectors, call this β.

$$\vec{x}_\beta = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_\beta$$

$$\underbrace{A}_{}\vec{x}_\beta = A(c_1\vec{v}_1 + \dots c_n\vec{v}_n) = c_1(A\vec{v}_1) + \dots + c_n(A\vec{v}_n)$$

$$= c_1\lambda_1\vec{v}_1 + \dots + c_n\lambda_n\vec{v}_n$$

$$= \begin{pmatrix} \underline{\lambda_1} c_1 \\ \underline{\lambda_2} c_2 \\ \vdots \\ \underline{\lambda_n} c_n \end{pmatrix}_\beta$$

A in β-word.

$$A\vec{x}_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_\beta$$

How did A act on β-coordinates

- Complex eigenvalues.

$$\det(A - \lambda I) = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1 \lambda + c_0$$

$$= c_n(\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

oh $\lambda_1 \dots \lambda_n$ are my eigenvalues.

But not every polynomial is factorable over $\mathbb{R}$.

E.G. $\lambda^2 + 1$ does not factor into real linear terms! (λ^2+1 is irreducible in $\mathbb{R}[\lambda]$)

optional

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \det\begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = (-\lambda)^2 - (-1)(1)$$

$$= \lambda^2 + 1$$

In fact $\det(A - \lambda I) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$

$$(\lambda - i)(\lambda + i) = \lambda^2 - \cancel{i\lambda} + \cancel{i\lambda} + (-i)(i) = \lambda^2 + -i^2$$
$$= \lambda^2 + 1$$

If λ is not a real number, what does that mean for
$$V_\lambda = \{ \vec{v} \in \mathbb{R}^n \mid Av = \lambda v \} \ ?$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad V_i = \text{eigenspace for } \lambda = i$$
$$= \{ v \in \mathbb{R}^2 \mid \underbrace{Av}_{\text{real}} = \underbrace{iv}_{\text{complex}} \} \quad !!$$

One way to fix this is to work in $\mathbb{C}^n$ - $\mathbb{R}^n$ but w/ $\mathbb{C}$
$$= \{ \vec{z} = (z_1, \ldots, z_n) \ \ z_i \in \mathbb{C} \}$$

Thm A is diagonalizable iff a basis of eigenvectors in $\mathbb{C}^n$ iff $\dim(V_\lambda) =$ alg mult

$$\dim(V_\lambda) \subseteq \mathbb{C}^n$$

Ex

$$A = \begin{pmatrix} -1 & 0 & -2 & 0 \\ 5 & 1 & 4 & 0 \\ 1 & 0 & 1 & 0 \\ -10 & 0 & -8 & 1 \end{pmatrix}$$

Find the eigenvalues and eigenspaces.

Not diagonalizable over $\mathbb{R}^4$!

But it works over $\mathbb{C}^4$.

$$\det(A - \lambda I) = \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1$$

(remember long division)

$$= (\lambda-1)^2(\lambda^2+1)$$

$\lambda = 1, 1$

alg mult is 2

$\lambda = i, -i$

each have alg mult $= 1$

$$V_i = \ker(A - iI) = \operatorname{span}\begin{pmatrix} 5-i \\ -13 \\ -3-2i \\ 26 \end{pmatrix}$$

$\dim(V_i) = 1$ = geom mult

alg mult = 1

$$V_{-i} = \ker(A + iI) = \operatorname{span}\begin{pmatrix} 5+i \\ -13 \\ -3+2i \\ 26 \end{pmatrix}$$

$\dim(V_{-i}) = 1$

alg mult = 1

$$V_1 = \ker(A - 1I) = \operatorname{span}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\dim(V_1) = 2$

= alg mult

eigenvectors

A is diagonalizable!

$$S = \begin{pmatrix} 5-i & 5+i & 0 & 0 \\ -13 & -13 & 1 & 0 \\ -3-2i & -3+2i & 0 & 0 \\ 26 & 26 & 0 & 1 \end{pmatrix}$$

$$\frac{i}{1} \quad \frac{-i}{1} \quad \frac{1 \quad 1}{2}$$

Ex $\boxed{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$ $\begin{pmatrix} \text{geom mult} \\ \lambda = 0 \end{pmatrix} = 1$

But $\det(A - \lambda I) = -\lambda^3$

$\lambda = 0, 0, 0$

$$\begin{pmatrix} \\ 0 \quad 0 \quad 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Interesting Facts

Prop A is not invertible iff $\lambda = 0$ is an eigenvalue.

Pf Suppose $\lambda = 0$ is an eigenvalue of A.

$$\iff V_0 = \left\{ \begin{array}{l} A\vec{v} = 0\vec{v} \\ A\vec{v} = 0 \end{array} \right\} \neq \{\vec{0}\}$$

$$V_0 = \ker(A) \neq \vec{0}$$

$$\iff A^{-1} \text{ not existing} \iff A \not\to I$$

A cannot row reduce to I.

Thm Let A be an $n \times n$ matrix.

Then 1) $\det(A) = \lambda_1 \cdots \lambda_n$

2) $tr(A) = \sum \lambda_i = \lambda_1 + \lambda_2 + \dots + \lambda_n$

3) A is positive definite iff $\lambda_i > 0$ for all λ_i.

Def Let A be an $n \times n$ matrix.

$$\operatorname{tr}(A) = \sum a_{ii}$$

$$= a_{11} + a_{22} + \dots + a_{nn}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & \cdot & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$\operatorname{tr}\begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4 \quad (\text{symmetries of shapes})$$

$$= \lambda_1 + \lambda_2 + \lambda_3 \quad \text{somehow??}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(A) = 0 \cdot 0 - (-1) \cdot 1 = \boxed{1}$$

$$\lambda_1 = i, \quad \lambda_2 = -i$$

$$\lambda_1 \lambda_2 = i(-i)$$
$$= -i^2$$
$$= -(-1) = \boxed{1}$$
$$= \det$$

$$\operatorname{tr}(A) = 0 + 0 = \boxed{0}$$

$$\lambda_1 + \lambda_2 = i + (-i) = \boxed{0}$$

Pf $\det(A - \lambda I) = \boxed{c_n}\lambda^n + \boxed{c_{n-1}}\lambda^{n-1} + \cdots + c_1\lambda + c_0$

possibly complex roots $= \boxed{c_n}(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

What is the coefficient in front of λ^n?

$$\det(A - \lambda I) = \det\begin{pmatrix} a_{11} - \lambda & & a_{ji} & \\ & a_{22} - \lambda & & \\ a_{ij} & & \ddots & \\ & & & a_{nn} - \lambda \end{pmatrix}$$

$$= \underbrace{(a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda)}_{n \text{ copies of } -\lambda} + \cdots \lambda^{n-2}$$

less λ's

FOIL $= \underbrace{(-1)^n}_{c_n}\lambda^n + \underbrace{(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})}_{c_{n-1} = (-1)^{n-1}\operatorname{tr}(A)}\lambda^{n-1} + \cdots + c_0$

$$\det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1} \boxed{(a_{11} + a_{22} + \dots + a_{nn})} \lambda^{n-1} + \dots + c_0$$

$$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$

$$= (-1)^n \lambda^n + (-1)^{n-1} \boxed{(\lambda_1 + \dots + \lambda_n)} \lambda^{n-1} + \dots + \lambda_1 \dots \lambda_n$$

So equating coefficients of λ^{n-1}

$$\lambda_1 + \dots + \lambda_n = a_{11} + \dots + a_{nn} = \operatorname{tr}(A)$$

HW 10 posted!

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ Find a matrix A in $v_1 v_2 v_3$ - coordinates.

$$\underline{B} = (v_1\ v_2\ v_3)^{-1} A (v_1\ v_2\ v_3)$$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 4y \\ -2x + 3y \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{B = S^{-1} \boxed{AS}}$$

(L) $$B = \underline{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}} \underline{\boxed{\begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}}$$

$$\begin{pmatrix} -3 & -5 \\ 1 & 5 \end{pmatrix}$$

$$L\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Complete eigenvalue

geom mult = alg mult

$$\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$\det\begin{pmatrix} -1-\lambda & 2 \\ 3 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda) - 6$$

$$= \lambda^2 - \lambda + \lambda - 1 - 6$$

$$= \lambda^2 - 7 = 0$$

$$\lambda = \sqrt{7}, \ -\sqrt{7}$$

alg mult $= 1$ 1

geom mult $= \dim(V_\lambda)$

$$V_{\sqrt{7}} = \ker(A - \sqrt{7}I)$$

$$= \text{span}\begin{pmatrix} 1+\sqrt{7} \\ 2 \end{pmatrix}$$

$$\dim(V_{\sqrt{7}}) = 1$$

$\lambda = \sqrt{7}$ is complete because it has the same alg mult and geom mult.