Prop $\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$ where the λ are the eigenvalues.

Pf $\det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n-1} \underbrace{tr(A)} \lambda^{n-1} + \cdots$

$\cdots + c_1 \lambda^1 + \boxed{c_0}$ ← compute!

$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

$= (-1)^n \lambda^n + (-1)^{n-1} \underbrace{(\lambda_1 + \cdots + \lambda_n)} \lambda^{n-1}$

$+ \cdots + \cancel{(-1)^n (-1)^n}_{1} \boxed{\lambda_1 \lambda_2 \cdots \lambda_n}$

$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

In general

$p(x) = a_n x^n + \dots + a_1 x^1 + a_0$

$p(0) = a_n \cdot 0^n + a_{n-1} \cdot 0^{n-1} + \dots$
$\dots + a_1 \cdot 0 + a_0 = a_0$

Plug in $\lambda = 0$

$$\det(A - \cancel{0I}) = c_0 = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\det(A) = c_0 = \lambda_1 \lambda_2 \cdots \lambda_n$$

$\square$

Def: An eigenvalue is called <u>complete</u> if alg mult = geom mult.

$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$

$\lambda_1 = \sqrt{7}$ alg = geom = 1

$\lambda_2 = -\sqrt{7}$ alg = geom = 1

both <u>complete</u>!

It's not diagonalizable

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda = 0, 0, 0$

alg mult = 3

but $V_0 = \ker(A) = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

geom mult = 1 <u>not complete</u>

<u>Thm</u> Let A be a symmetric matrix. $(A = A^T)$

(a) All eigenvalues of A are real. $(\lambda \in \mathbb{R})$

(b) If λ, μ are distinct eigenvalues $(\lambda \neq \mu)$

then $V_\lambda \perp V_\mu$. $(\vec{v}_\lambda \cdot \vec{v}_\mu = 0)$

$\vec{v}_\lambda \perp \vec{v}_\mu$

(c) All symmetric matrices have an <u>orthonormal</u> basis of eigenvectors.

(d) $A = Q \Lambda Q^T$ where Q is orthogonal

Λ diagonal matrix of eigenvalues

Spectral decomposition

Pf (a) If $A = A^T$, then $\lambda \in \mathbb{R}$.

Step 1) If A is symmetric $(A\vec{v}) \cdot \vec{w} = \vec{v} \cdot (A\vec{w})$.

$$(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T \vec{w} = \bar{v}^T A^T w = \bar{v}^T A w = \bar{v}^T (Aw)$$
$$= v \cdot (Aw)$$

symmetry!

v, w maybe complex!

Step 2 If v is an e.genvector

$$Av \cdot v = \lambda v \cdot v = \lambda (v \cdot v) = \lambda \|v\|^2$$
$$v \cdot (Av) = v \cdot \lambda v = \bar{\lambda} (v \cdot v) = \bar{\lambda} \|v\|^2 \qquad (3.6)$$

$$\lambda \|v\|^2 = \bar{\lambda} \|v\|^2 \implies \lambda = \bar{\lambda}$$
$$\implies \lambda \in \mathbb{R}$$

$\lambda = x + iy$

$\bar{\lambda} = x - iy$

$\lambda = \bar{\lambda} = x$

$\in \mathbb{R}$

Quite optional to know

(b) If λ, μ are distinct eigenvalues $(\lambda \neq \mu)$

then $V_\lambda \perp V_\mu$. $\quad (\vec{v}_\lambda \cdot \vec{v}_\mu = 0)$

$\vec{v}_\lambda \perp \vec{v}_\mu$

<u>Pf</u> Suppose $\lambda \neq \mu$. Let $v \in V_\lambda$, $w \in V_\mu$.

$$\lambda(v \cdot w) = \lambda v \cdot w = Av \cdot w = v \cdot Aw$$

$$= v \cdot \mu w = \mu(v \cdot w)$$

All in all $\lambda(v \cdot w) = \mu(v \cdot w)$

$$(\lambda - \mu)(v \cdot w) = 0 \qquad \lambda - \mu \neq 0$$

$$\Rightarrow v \cdot w = 0 \qquad v \perp w$$

and $V_\lambda \perp V_\mu$.

(c) All symmetric matrices have an orthonormal basis of eigenvectors.

$\lambda_1, \ldots, \lambda_n$ w/ repeats

$\lambda_1, \ldots, \lambda_k$ k distinct eigenvalues

know this!

$V_{\lambda_1} \perp V_{\lambda_2} \perp \ldots \perp V_{\lambda_k}$ do G-S on each of them individually

Pf Let v_1 be an eigenvector.

Consider $W = \mathrm{span}(v_1)^{\perp}$ $(\dim(W) = n-1)$

$A|_W$ is still symmetric. By induction W has orthonormal basis of eigenvectors $u_2, \ldots, u_n$

restrict

Super optimized to read.

$\Longrightarrow \frac{v_1}{\|v_1\|}, u_2, \ldots, u_n$ $\square$

(d) $A = Q \Lambda Q^T$.

We know from part (c) that A is diagonalizable.

Pick $u_1 \dots u_n$.

Then $Q = (\vec{u}_1 \dots \vec{u}_n)$

Diagonalization $\Lambda = Q^{-1} A Q = Q^T A Q$

$\Longrightarrow A = Q \Lambda Q^T$ $\square$

know this!

$V_{\lambda_1} \perp V_{\lambda_2} \perp \dots \perp V_{\lambda_k}$ do G-S on each of them individually

Again to find the orthonormal basis of eigenvectors of A symmetric matrix, find the eigenspaces and G-S on each of them.

Corollary A matrix K is pos def iff all of its eigenvalues are $\lambda > 0$.

Pf use part (c).

Thm K is pos def iff all of its pivots are positive!

Ex Find an orthonormal basis of eigenvectors for the matrix

$$A = \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}.$$

Remember find V_{λ_i} usual way, then do G-S.

$$\det(A - \lambda I) = -\lambda^3 + 23\lambda^2 - 150\lambda + 216$$

How do we find the roots?

$$\det(A) = c_0 = \boxed{216 = \lambda_1 \lambda_2 \lambda_3}$$

If $\lambda_1, \lambda_2, \lambda_3$ are integers then we have a finite amount of possibilities.

$$216 = 2^3 3^3 = 8 \cdot 27$$

$$-\lambda^3 + 23\lambda^2 - 150\lambda + 216$$

$$= -(\lambda - 2)(\lambda^2 - 21\lambda + 108)$$

$$= -(\lambda - 2)(\lambda - 9)(\lambda - 12)$$

$\pm$

	2^0	2^1	2^2	2^3
3^0	1	$\boxed{2}$	4	8
3^1	3	6	$\boxed{12}$	24
3^2	$\boxed{9}$	18	36	72
3^3	27	54	108	216

Integer Possibilities for λ!

$\lambda_1 = 2$

$\lambda_2 = 9$

$\lambda_3 = 12$

All complete!

since alg mult = 1

$$V_{\lambda=2} = \ker(A - 2I) = \text{span}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad u_1 = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$$

$$V_{\lambda=9} = \ker(A - 9I) = \text{span}\begin{pmatrix}-1\\1\\1\end{pmatrix} \qquad u_2 = \begin{pmatrix}\frac{-1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$$

$$V_{\lambda=12} = \ker(A - 12I) = \text{span}\begin{pmatrix}1\\-1\\2\end{pmatrix} \qquad u_3 = \begin{pmatrix}\frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{pmatrix}$$

Thm predicts that these are orthogonal to each other!

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\1\end{pmatrix} = -1 + 1 = 0 \quad ✓$$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = 2 - 2 = 0 \quad ✓$$

$$\begin{pmatrix}-1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = -1 - 1 + 2 = 0 \quad ✓$$

As predicted!

A $\qquad \lambda = 1 \qquad \lambda = 2,2 \qquad$ Repeated roots $\Rightarrow$ G-S

$$V_{\lambda=1} = \text{Span}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$V_{\lambda=2,2} = \text{span}\begin{pmatrix}-3\\3\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$$

* $\begin{pmatrix}-3\\3\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}$ need not be orthogonal

$\begin{pmatrix}-3\\3\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}$ is not orthogonal basis

$$(2-\lambda)(-7-\lambda) - 36 \qquad \begin{pmatrix}2 & 6\\6 & -7\end{pmatrix}$$

$$-14 + 7\lambda - 2\lambda + \lambda^2 - 36$$

$$\lambda^2 + 5\lambda - 50 = (\lambda - 5)(\lambda + 10) \qquad \begin{pmatrix}-3 & 6\\6 & -12\end{pmatrix}$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\downarrow$$

$$u_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\|v_2\| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + 1^2}$$

$$= \sqrt{\tfrac{1}{4} + 1} = \sqrt{\tfrac{5}{4}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{4}{5}} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$= \frac{2}{\sqrt{5}} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$2 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} \longrightarrow \cancel{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$