Matrix Multiplication, Transposes, Inverses

(1.2) (1.6) (1.5)

m rows

n columns

Def A matrix is an $m \times n$ grid of numbers

(real numbers, (0, 1, π, $\frac{1}{5}$, -37.2, e,)) (i is imaginary)

$$A = \begin{array}{c} \text{1st row} \\ \text{2nd} \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & & \ddots & & & \\ \vdots & & & & & \\ a_{m1} & \cdots & \cdots & & & a_{mn} \end{bmatrix}$$

$A = (A)_{ij}$, $1 \le i \le m$, $1 \le j \le n$

$(A)_{ij}$ = entry in the i^{th} row and j^{th} column

$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & \boxed{\pi} \end{bmatrix}$

2 × 3 matrix
row columns

$B = \begin{bmatrix} 3 & -2 \\ \boxed{1} & 0 \\ -1 & 0 \end{bmatrix}$

3 × 2
row columns

$(A)_{23} = \pi$

$(B)_{21} = 1$

You can multiply matrices!

Let A be a $m \times n$ matrix. B is an $n \times p$ matrix.

Then AB is an $m \times p$ matrix with entries

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

$$(AB)_{ij} = \underbrace{a_{i1} b_{1j}}_{k=1} + \underbrace{a_{i2} b_{2j}}_{k=2} + \underbrace{\cdots}_{\cdots} + \underbrace{a_{in} b_{nj}}_{k=n}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\underset{m \times n}{\begin{bmatrix} — a_{1*} — \\ — a_{2*} — \\ \vdots \\ — a_{n*} — \end{bmatrix}} \overset{n \times p}{\begin{bmatrix} | & | & & | \\ b_{*1} & b_{*2} & \cdots & b_{*n} \\ | & | & & | \end{bmatrix}}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 1 & \quad & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ \end{array} & \underset{m \times p}{\begin{bmatrix} \boxed{} & & \\ & & \boxed{} \\ & & \\ \end{bmatrix}} \end{array}$$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & \boxed{\pi} \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 \\ \boxed{1} & 0 \\ -1 & 0 \end{bmatrix} \qquad AB$$

$2 \times \boxed{3}$ $\qquad$ $\boxed{3} \times 2$

2×2 matrix

$$AB = \begin{bmatrix} \boxed{8} & \boxed{-6} \\ -\pi - 1 & 0 \end{bmatrix}$$

$3 \cdot 3 + 1 \cdot 1 + 2 \cdot (-1) = 8$

$3 \cdot (-2) + 1 \cdot 0 + 2 \cdot 0 = -6$

$0 \cdot 3 + (-1) \cdot 1 + \pi\,(-1) = -\pi - 1$

$0 \cdot (-2) + -1 \cdot 0 + \pi \cdot 0 = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}$$

You can't multiply these!

$$AB = \begin{bmatrix} \square\ ??? & \\ & \end{bmatrix}$$

AB does not exist!

$3x + 1y = 2$
$-1x + 2y = 5$

$$\rightsquigarrow \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

Numbers		Matrix
$ax = b$	$\longleftrightarrow$	$A\vec{x} = \vec{b}$
$x = \frac{b}{a}$	$\longleftrightarrow$	$\vec{x} = \cancel{\frac{\vec{b}}{A}} = \underline{A^{-1}}\vec{b}.$
1	$\longleftrightarrow$	I

How do you make A^{-1}?
Define A^{-1}?

Def Let A be an $n \times n$ matrix.

Note: You can only invert a matrix w/ the same number of rows and columns!

Then A^{-1} is the unique matrix such that

$$AA^{-1} = A^{-1}A = I_n \qquad \left(a\frac{1}{a} = 1\right)$$

where $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$, called the identity matrix.

$(IB = B)$

$(BI = B)$

How do I know A^{-1} exist? Why is it unique?

Not every matrix has an inverse!

Ex

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has no inverse! ($\frac{1}{0}$ doesn't exist 0^{-1})

Why unique, if it exists?

Pf Let A be an invertible matrix. A^{-1} and M are both inverses to A. $AA^{-1} = I$, $AM = I$.

$$A^{-1} = A^{-1}I = A^{-1}AM = IM = M.$$

Therefore $A^{-1} = M$ all along! So A^{-1} is unique.

Warning! Matrix Multiplication is not commutative!

A.K.A You can't switch the order of multiplication.

$$AB \neq BA$$

$$\underset{3\times 2}{A} \quad \underset{2\times 3}{B} \qquad \underset{3\times 3}{AB} \neq \underset{2\times 2}{BA}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$\neq$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This is how you compute the inverse for a 2×2.

Ex

$$C = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \qquad C^{-1} = \frac{1}{(-1)(0)-(3)(2)}\begin{bmatrix} 0 & -2 \\ -3 & -1 \end{bmatrix}$$

$CC^{-1} = I$ should happen!

$$\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}\frac{1}{-6}\begin{bmatrix} 0 & -2 \\ -3 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$CC^{-1} = I$ should happen!

$$\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \frac{1}{-6} \begin{bmatrix} 0 & -2 \\ -3 & -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$C^{-1}C = \frac{1}{6} \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

All the time $AA^{-1} = I = A^{-1}A$.

So A, A^{-1} commute.

Transposes (rows vs. columns)

Let A be $m \times n$ matrix. $A = (A)_{ij}$

Then we define $A^T = (A^T)_{ij} = (A)_{ji}$

aka A^T is $n \times m$ and A^T is formed
swapped

by turning columns into rows.

Ex

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 2 & 5 \end{bmatrix}$$

2×3 $\qquad$ 3×2

Formulas:

- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^T = B^T A^T$ (notice order is flipped!)
- $(AB)^{-1} = B^{-1} A^{-1}$

$$I = (AB)(AB)^{-1} = AB\cancel{A^{-1}B^{-1}}$$ can't switch order

$$= A\underline{BB^{-1}}A^{-1} = AA^{-1} = I$$

Def A $n \times n$ matrix is called symmetric if $A = A^T$.

Ex $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} = A^T$.

Rotation matrix $R = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$

Algebra

multiplying, polynomials

CS algorithms

Analysis

derivatives, integrals,

engineers

Differential Equations