Posted HW: 1 due Friday

Row Reduction, Pivots, Computing Inverses, Permutations

new?

Solving Linear Systems

$$\begin{aligned} x + y + (-3)z &= \boxed{1} \\ 2x - y + 2z &= \boxed{0} \\ 3x \quad\ - 2z &= \boxed{1} \end{aligned}$$

turn into $\rightsquigarrow$ augmented matrix

$$\begin{pmatrix} 1 & 1 & -3 & \vdots & 1 \\ 2 & -1 & 2 & \vdots & 0 \\ 3 & 0 & -2 & \vdots & 1 \end{pmatrix}$$

*

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ \boxed{2} & -1 & 2 & 0 \\ 3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{-2r_1 + r_2 = r_2'} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & -3 & 8 & -2 \\ \boxed{3} & 0 & -2 & 1 \end{array}\right) \xrightarrow{-3r_1 + r_3 = r_3'} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & -3 & 8 & -2 \\ 0 & \boxed{-3} & 7 & -2 \end{array}\right) \xrightarrow{-r_2 + r_3 = r_3'}$$

$-r_3 + r_2 = r_3'$

oh but confusing to me

In row reduction, 3 basic operations

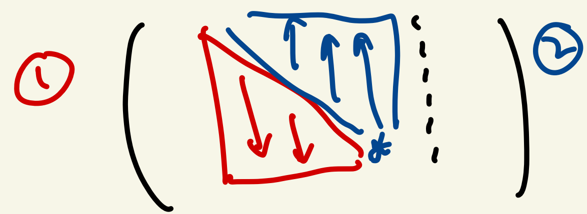
* ① $\underbrace{r_j'}_{\text{new}} = c r_i + \underbrace{r_j}_{\text{old}}$

later ② switch two rows

$r_i \longleftrightarrow r_j$

* ③ $r_i' = c r_i$

multiply ith row by a constant



$$\left(\begin{array}{ccc:c} \boxed{1} & 1 & -3 & 1 \\ 0 & \boxed{-3} & 8 & -2 \\ 0 & 0 & \boxed{-1} & 0 \end{array}\right)$$

$\ne 0$

This matrix is an example of an upper triangular matrix.

$$x + y - 3z = 1$$
$$-3y + 8z = -2$$
$$-z = 0$$

$x + \frac{2}{3} - 0 = 1$

$x = \frac{1}{3}$

$-3y + 0 = -2$

$y = \frac{2}{3}$

$z = 0$

During the 1st step where you cancel the bottom left triangle, it's best only do row operations $cr_i + r_j$ $i < j$. multiply the constant by the row on top.

Gaussian Elimination, (downsweep)

These entries are also important, they are called pivots.

3 nonzero pivots $\longrightarrow$ unique solution

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -3 & 1 \\ 0 & \boxed{-3} & 8 & -2 \\ 0 & 0 & \boxed{-1} & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & -3 & 8 & -2 \\ 0 & 0 & -1 & 0 \end{array}\right) \quad *$$

$$\xrightarrow{-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 0 & -3 & \boxed{8} & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-8r_3+r_2} \left(\begin{array}{ccc|c} 1 & 1 & \boxed{-3} & 1 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{3r_3+r_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & \boxed{-3} & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}r_2} \left(\begin{array}{ccc|c} 1 & \boxed{1} & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-r_2+r_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} x = \frac{1}{3} \\ y = \frac{2}{3} \\ z = 0 \end{array}$$

Actually these 8 steps row reduce the original matrix

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3 pivots $\downarrow$ 3 1's.

Ticket to calculating matrix inverses.

① $r_j' = cr_i + r_j$ $\rightsquigarrow$ $\begin{array}{cc} & \\ j & \begin{pmatrix} 1 & & & \\ & 1 & & \\ c & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \\ & i \end{array}$ Elementary matrix

② $r_i \longleftrightarrow r_j$

③ $r_i' = cr_i$

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \xRightarrow{-2r_1+r_2} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 3 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 3 & 0 & -2 \end{pmatrix}$$

An elementary matrix is a matrix E such that EA is the same as applying a row operation to A.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is elementary $\rightsquigarrow$ $-2r_1 + r_2$

① $r_j' = cr_i + r_j$

② $r_i \longleftrightarrow r_j$

③ $r_i' = cr_i$

①
$$\begin{array}{c|cccc} & i & & & \\ \hline & & & & \\ & \multicolumn{4}{c}{\left(\begin{array}{cccc} 1 & & & \\ & 1 & & \\ c & & 1 & \\ & & & 1 \end{array}\right)} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} i & & & \end{array} \\ \begin{array}{c} \\ \\ j \\ \\ \end{array} & \left(\begin{array}{cccc} 1 & & & \\ & 1 & & \\ c & & 1 & \\ & & & 1 \end{array}\right) \end{array}$$

②
$$\begin{array}{cc} & \begin{array}{cccc} & j & i & \end{array} \\ \begin{array}{c} \\ j \\ i \\ \\ \end{array} & \left(\begin{array}{cccc} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{array}\right) \end{array}$$

swap i^{th} j^{th} rows in I

③
$$\begin{array}{cc} & \begin{array}{ccccc} & & i & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} & \left(\begin{array}{ccccc} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) \end{array}$$

* $-2r_1 + r_2$

* $-3r_1 + r_3$

$-r_2 + r_3$

$-r_3$

$-8r_3 + r_2$

$3r_3 + r_1$

$-\frac{1}{3} r_2$

$-1 r_2 + r_1$

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

① $$\begin{array}{c|c} & \begin{array}{cccc} & i & & \end{array} \\ j & \begin{pmatrix} 1 & & & \\ & 1 & & \\ c & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

② $$\begin{array}{c|c} & \begin{array}{cccc} & j & i & \end{array} \\ \begin{array}{c} \\ j \\ i \\ \\ \end{array} & \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & \ddots \; 0 & \\ & & & 1 \end{pmatrix} \end{array}$$

swap i^{th} j^{th} rows in I

③ $$\begin{array}{c|c} & \begin{array}{cccc} & & i & \end{array} \\ i & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & c & \\ & & & 1 \\ & & & & 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \overset{*}{\begin{pmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{pmatrix}} \overset{*}{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} * \; -2r_1 + r_2 \\ * \; -3r_1 + r_3 \\ -r_2 + r_3 \end{array}\right\}$$

$\boxed{\text{LU decomp.}}$

$-r_3$

$-8r_3 + r_2$

$3r_3 + r_1$

$-\frac{1}{3} r_2$

$-1r_2 + r_1$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \overset{8}{\begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix}} \overset{7}{\begin{pmatrix} 1 & & \\ & -\frac{1}{3} & \\ & & 1 \end{pmatrix}} \overset{6}{\begin{pmatrix} 1 & & 3 \\ & 1 & \\ & & 1 \end{pmatrix}} \overset{5}{\begin{pmatrix} 1 & & \\ & 1 & -8 \\ & & 1 \end{pmatrix}}$$

$$\overset{4}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}} \overset{3}{\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}} \overset{*}{\begin{pmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{pmatrix}} \overset{*}{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}$$

product of elementary matrices is the inverse!

$* \; AX = I$ Solve for $X = A^{-1}$

$$\begin{cases} E_8 \cdots E_1 A X = E_8 \cdots E_1 I \\ A^{-1} = X = \underbrace{E_8 \cdots}\, E_1 I \end{cases}$$

doing row ops to I calculates A^{-1}

Algorithm for computing A^{-1}.

① $* \left(A \;\vdots\; I \right)$

② Row reduce A to be the identity matrix now I will be on the left.

③ Apply row ops to I on the right as well

$$\left\{ \left(\begin{array}{c:c} I & A^{-1} \end{array} \right) \right.$$ will be the end result.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 3 & 0 & -2 & 0 & 0 & 1 \end{array} \right) \xrightarrow[\text{8 row ops}]{\text{apply}} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 2/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & 10/3 & 7/3 & -8/3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right)$$

z free

∞, no solutions

Ex

$$\left(\begin{array}{ccc:c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

∞

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \end{array} \right)$$

$x + 2z = 0$ z free

$y + z = 1$

$y = -z$

$x = -2z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2z \\ -z \\ z \end{pmatrix} = \underline{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} z}$$

$$A^{-1} = \frac{1}{\det A} \boxed{A^{+}} \rightsquigarrow \text{terrible formula}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>works</u> !

A^{-1} doesn't exist if <u>$\det A = 0$</u>

$-r_1 + r_4 = r_4'$ is always

changes 4^{th} row

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & & & 1 \end{pmatrix}$$

$$r_j' = cr_i + r_j$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow[r_3']{-3r_1 + r_3} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 5 & -5 \end{pmatrix}$$

* doing

$$\begin{array}{c} \text{column} \quad \text{row} \\ -3r_1 + r_3 \end{array} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

*

elementary matrix

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 5 & -5 \end{pmatrix}$$