HW 1: due Friday 9/18

Office Hours: Tomorrow, Th 9/17 12:00 - 3:00

in this zoom

Last Time: Row reduction $\longrightarrow$ LU decomposition

Let A be a square matrix with
- all pivots nonzero (we can do backsubstitution)
- no row swaps need to be done when row reducing

then $A = LU$ where L is lower triangular w/ 1's on diagonal, U is upper triangular w/ nonzero diagonal

An upper triangular matrix is a matrix such that all entries below the diagonal are 0.

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \text{ is upper triangular}$$

$$(U)_{ij} = 0 \quad i > j$$

Ex
$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3 →

$$\begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

not in LU decomp.

A matrix is lower triangular if all entries above the diagonal are 0.

$$(L)_{ij} = 0 \quad j > i$$

Ex
$$\begin{pmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

How do we compute $A = LU$?

start $= A$

⓪ $\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$

$= U$

③ $\begin{pmatrix} \boxed{1} & 1 & -3 \\ 0 & \boxed{-3} & 8 \\ 0 & 0 & \boxed{-1} \end{pmatrix}$

has nonzero entries because A has 3 pivots

⑧ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solved!

these coefficients make L

$\left.\begin{array}{l} -2r_1 + r_2 \\ -3r_1 + r_2 \\ -r_2 + r_3 \end{array}\right\}$ regular Gaussian elimination (downsweep)

$\left.\begin{array}{l} -r_3 \\ -8r_3 + r_2 \\ 3r_3 + r_1 \\ -\frac{1}{3}r_2 \\ -1r_2 + r_1 \end{array}\right\}$ backsubstitution

For $A = LU$ all you need is this phase of the row reduction!

Given a matrix A, the U is just the resulting after doing the GE on A.

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}}_{①*} \underbrace{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{pmatrix}}_{②*} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{③*} \begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$$

blank entries mean 0

$$U \qquad\qquad\qquad\qquad A$$

$$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{pmatrix}$$

$$U = L^{-1} A$$

$$A = LU$$

upper Δ after phase 1

original matrix

actually tells you which row operations to do.

the negatives to the constants from row operation go in L! why?

$$A = \underbrace{\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}}_{E_1^{-1}} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{pmatrix}}_{E_2^{-1}} \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}}_{\text{why? } E_3^{-1}} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{matrix} \\ -2 \\ -3 \end{matrix} \begin{pmatrix} \boxed{1} & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & \boxed{1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -3 \\ & -3 & 8 \\ & & -1 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad U$$

Given an elementary matrix corresponding to

$$r_2' = -2r_1 + r_2 \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix}^{-1} \longleftrightarrow$$ reverse the row operation! $r_2' = 2r_1 + r_2$

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \xrightarrow[r_2' = -2r_1 + r_2]{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 3 & 0 & -2 \end{pmatrix}$$

$r_2' = 2r_1 + r_2$

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

This is why in L, the entries are the negatives!

L actually encodes how to row reduce U backwards to A.

Problem Find the the LU decomposition of $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{-2r_1 + r_3} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & -2 & -5 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix} = U$$

$-r_1 + r_2$
1 column
2 row

$2r_1 + r_3$
1 column
3 row

$\begin{pmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{pmatrix}$

$-r_2 + r_3$
2 column
3 row

-1 in
2 column
3 row

$\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}$

always L has 1's on diag.

3 pivots!

We can do back sub

$$A = \underbrace{\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix}}_{U}$$

Done!

Make sure how lower bottom triangle in the right order!

Why? Within math, how matrices behave among all other matrices, tells how how lower matrices can combine to give you most matrices!

Computer algorithms for solving linear systems.

<u>Ex</u>

$$A \rightarrow \begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & -3 & 1 \\ 2 & -1 & 2 & 0 \\ 3 & 0 & -2 & 1 \end{array}\right)$$

Compute LU decomp of A

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & \boxed{1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -3 \\ & -3 & 8 \\ & & -1 \end{pmatrix}$$

3 steps instead of 8.

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 & -3 \\ & -3 & 8 \\ & & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Since L is lower triangular, this is easy to solve!

$a = 1$

$2a + b = 0 \longrightarrow 2 + b = 0 \quad b = -2$

$3a + b + c = 1 \longrightarrow 3 - 2 + c = 1$

$c = 0$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & \boxed{-3} & 8 \\ 0 & 0 & \boxed{-1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

can't be 0!

Since U is upper Δ, also fast!

$$x + y - 3z = 1 \longrightarrow x + \frac{2}{3} = 1 \qquad x = \frac{1}{3}$$

$$-3y + 8z = -2 \longrightarrow -3y = -2 \qquad y = \frac{2}{3}$$

$$-z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \end{pmatrix}$$

This is fast for a computer compared to doing 5 other row ops.

Next row swaps, permutations in general.

$$\begin{pmatrix}1&1&-3\\2&-1&2\\3&0&-2\end{pmatrix} \xrightarrow{-2r_1+r_2} \xrightarrow{-3r_1+r_3} \begin{pmatrix}1&1&-3\\0&-3&8\\0&-3&7\end{pmatrix}$$

$$\Big\downarrow -r_2+r_3$$