If A is an $n \times n$ matrix

- A n pivots (U has all nonzero diagonal entries)

- no ~~row~~ swapping or $r_i' = cr_i$ operations

$$(\text{only } r_j' = cr_i + r_j)$$

then $A = LU$.

what if we had row swapping?

$$A = \begin{pmatrix} 0 & 0 & 2 \\ \boxed{1} & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

No way to cancel $1 = A_{21}$ or $2 = A_{31}$ w/out doing a row swap!

In this case A has a <u>permuted</u> LU decomposition.

$$\underbrace{P}_{\substack{\text{permutation}\\ \text{matrix} \\ ?}} A = \underbrace{L}_{\substack{\text{unilower}\\ \text{triangular}}}\underbrace{U}_{\text{upper triangular}}$$

We need to know how permutations work.

Def: A permutation on n objects is a way to reorder those n objects. (A permutation is a bijection from $\{1, 2, \dots, n\}$ to itself.) optional def

Ex $n=2$ There are 2 ways to rearrange these.

$$\begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 2 \end{array}$$

id

$$\begin{tikzcd} 1 \arrow[dr] & 1 \\ 2 \arrow[ur] & 2 \end{tikzcd}$$

swapping 1 and 2

(12)

$$\begin{array}{l} 1 \longrightarrow 2 \\ 2 \longrightarrow 1 \end{array}$$

$$\begin{array}{cc} 1 & 2 \\ \hline 2 & 1 \end{array}$$

What if $n = 3$? There are 6 permutations of 3 objects.

$1 \longrightarrow 1$
$2 \longrightarrow 2$
$3 \longrightarrow 3$
id

$1 \longrightarrow 2$
$2 \longrightarrow 1$
$3 \longrightarrow 3$
$\boxed{(12)}$
$\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 1 & 3 \end{array}$

$1 \longrightarrow 3$
$2 \longrightarrow 2$
$3 \longrightarrow 1$
(13)

$1 \longrightarrow 1$
$2 \longrightarrow 3$
$3 \longrightarrow 2$
(23)

$1 \longrightarrow 2$
$2 \longrightarrow 3$
$3 \longrightarrow 1$
(123)

$1 \longrightarrow 3$
$2 \longrightarrow 1$
$3 \longrightarrow 2$
$\boxed{(132)}$

All 6 permutations on 3 objects

(132) cycle notation

There are $n!$ permutations on n objects.

(24 perms on 4 objects e.g.)

Reason (n choices for where 1 goes)

$\times$ (n−1 choices for 2)

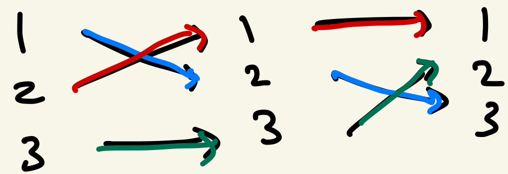
$\times$ (n−2 choices for 3)

$\vdots$

$\times$ (2 choices for n−1)

$\times$ (1 choice for n) $= n!$

You can compose permutations



$$(12) \circ (23)$$
$$\shortparallel$$
$$\begin{aligned} 1 &\to 3 \\ 2 &\to 1 \\ 3 &\to 2 \end{aligned} \qquad (132)$$

You might see $S_n = \{$ set of all permutations on n objects $\}$

Let's say we wanted a matrix P such that

$$P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$ (for example)

All P is doing is permuting x, y, z by (132)
$$\begin{aligned} 1 &\to 3 \\ 2 &\to 1 \\ 3 &\to 2 \end{aligned}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} \underline{0}x + \underline{0}y + \underline{1}z \\ 1x + 0y + 0z \\ 0x + 1y + 0z \end{pmatrix}$$

$$\uparrow$$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is the permutation matrix corresponding to $(1\,3\,2)$

$1 \longrightarrow 3$
$2 \longrightarrow 1$
$3 \rightarrow 2$

All permutations on n objects correspond to a matrix, called a permutation matrix, which just rearranges the components of a vector according to the permutation.

$1 \to 3$
$2 \to 1$
$3 \to 2$

$$\begin{array}{l} \text{3rd row of } I \\ \text{1st row of } I \\ \text{2nd row of } I \end{array} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{array}{l} \xleftarrow{1 \to 3} \\ \xleftarrow{2 \to 1} \\ \xleftarrow[3 \to 2]{} \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to make P, take I and rearrange rows of I by the permutation

$$\underset{\text{id}}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}, \quad \underset{(12)}{\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}}, \quad \underset{(13)}{\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}}, \quad \underset{(23)}{\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}}$$

$$\underset{(132)}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}, \quad \underset{(123)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

3×3 permutation matrices $\left\{ \begin{array}{l} \text{permutation} \\ \text{composition} \\ \text{becomes} \\ \text{matrix} \\ \text{multiplication} \end{array} \right\}$

In a permuted LU decomp $PA = LU$

P encodes all of the row swapping over the course of the row reduction

$r_j' = cr_i + r_j$

Let's say A has n pivots (nonsingular)

then it has a $PA = LU$ decomposition.

$$\begin{pmatrix} \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$ is singular

2 pivots < 3

Eg. $$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[\text{swap } r_1 r_2]{①} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[②]{-2r_1 + r_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & -7 & 2 \end{pmatrix}$$

$\xleftarrow{2r_1 + r_3}$

Since we swapped $r_1 r_2$, P becomes the permutation matrix for
$1 \to 2$
$2 \to 1$
$3 \to 3$

① $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$L = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

② $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$L = \begin{pmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{pmatrix}$

③ $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[\text{swap } r_1 r_2]{①} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[②]{-2r_1 + r_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & -7 & 2 \end{pmatrix} \xrightarrow{\text{swap } r_2 r_3} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$2r_1 + r_3$

$2r_1 + r_2$

So

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

is the permuted LU decomp.

A $PA = LU$ decomp is like writing the row swaps at the beginning and not throughout.

?

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[r_1 r_2]{\text{swap}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 2 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[r_2 r_3]{\text{swap}} \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow[-2r_1 + r_2]{} \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

(not r_3)

$$r_i' = c r_i \rightsquigarrow \begin{array}{c c} & i \\ i & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \end{array}$$

$$PA = LDU \qquad U = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

all constant scaling ups

$$U = \underline{\begin{pmatrix} 1 & & \\ & -\frac{1}{7} & \\ & & \frac{1}{2} \end{pmatrix}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -14 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$\uparrow$

P

$$A = P^{-1}LU$$

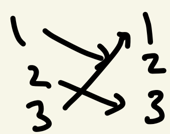
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

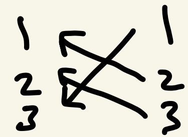
P^{-1} corresponds to backwards permutation

no need to do this for linear solving algorithm w/ $A = LU$

P

$1 \to 2$
$2 \to 3$
$3 \to 1$





$$P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} \xrightarrow[r_1 r_3]{\text{swap}} \begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_1 + r_2} \begin{pmatrix} 2 & -1 & 2 \\ 0 & \frac{7}{2} & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 1/2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ 0 & 7/2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

This is another permuted LU decomp.

It's not unique!

$$\overset{P}{\begin{pmatrix} & & \\ & & \end{pmatrix}}\overset{A}{\begin{pmatrix} & & \\ & & \end{pmatrix}} = \overset{L}{\begin{pmatrix} & & \\ & & \end{pmatrix}}\overset{U}{\begin{pmatrix} & & \\ & & \end{pmatrix}}$$

$AB = BA$ means A and B commute.

Which matrices can switch over w/ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

$$\xrightarrow[-2r_1+r_2]{} \quad \xrightarrow[-3r_1+r_3]{} \quad \xrightarrow[-1r_2+r_3]{}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix} A = U$$