__Def__ A __nonsingular__ matrix is a square matrix ($n \times n$) w/ n pivots. Recall that a pivot is a nonzero diagonal entry in the upper triangular matrix U in the permuted LU decomp.

Every example so far has been nonsingular.

$$\begin{bmatrix} \boxed{2} & 3 & 0 \\ & \boxed{-7} & 1 \\ & & \boxed{-1} \end{bmatrix}$$ 3 pivots.

Non example (singular matrix)

$$\begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ 2 pivots!

We can understand nonsingular matrices as follows.

Let A be nonsingular. Let's say $Ax=b$ is a linear system.

Then we know that A has n pivots. So we can now reduce

$$[A \,\vdots\, b] \longrightarrow [U \,\vdots\, b']$$

U is upper Δ w/ nonzero diagonal entries

$$\xrightarrow[\text{substitution}]{\text{back}} [I \,\vdots\, c]$$

c is the answer to $Ax=b$

i.e. $c = A^{-1}b$.

Furthermore

$$[A \,\vdots\, I] \longrightarrow [U \,\vdots\, M'] \xrightarrow[\text{sub.}]{\text{back}} [I \,\vdots\, A^{-1}]$$

So every nonsingular matrix is invertible!

Actually if A^{-1} exists then A has n pivots (i.e. A is nonsingular).

(I can explain a bit better by the end of chapter 2.)

Nonsingular and invertible are interchangable.

what do we do if A is not invertible?

Let $Ax = b$ be a system of eq'ns. But A^{-1} doesn't exist, so the system doesn't have a unique sol'n $\cancel{x = A^{-1}b}$.

We reduce A to reduced row echelon form to understand $Ax = b$.

Def We say a matrix M is in reduced row echelon form if all rows are either

leading 1

* (a) all zeroes

(b) start w/ some zeroes and then a $\boxed{1}$

such that the 1's will be the only nonzero entry in their column.

Pivots will turn into leading 1's

Furthermore all rows that are zero are at the bottom of the matrix.

All leadings 1's must go down and to the right.

$$M = \begin{bmatrix} \boxed{1} & 0 & 0 & * & * & * & * & * \\ 0 & \boxed{1} & 0 & * & * & * & * & \\ 0 & 0 & \boxed{1} & & & & & \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

rows of zeros

leading 1's

could be anything.

Ex

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots!

This is in reduced row echelon form RREF

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

is in RREF (easiest RREF matrix)

Not in RREF

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 4 & 2 \end{bmatrix}$$

This 3 needs to be cleared out! Use this one to clear out the 3.

Leading 1's should clear out their column.

$$\xrightarrow{-3r_2+r_1} \begin{bmatrix} \boxed{1} & 2 & 0 & 3 & -5 \\ 0 & 0 & \boxed{1} & -1 & 2 \end{bmatrix}$$ in RREF now!

leading 1's

We can RREF to solve noninvertible systems.

Ex

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 4 & -1 \\ -2 & 8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\downarrow$

I?

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ -2 & 8 & 1 & 0 \end{array}\right] \xrightarrow{-2r_1+r_3} \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-r_2+r_3} \left[\begin{array}{ccc|c} \boxed{-1} & 2 & 1 & 0 \\ 0 & \boxed{4} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2 pivots,

no back substitute

Infinite number of sol'ns!

Columns w/ pivots / leadings 1's correspond to dependent variables, else it's a free variable.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc} -1 & 2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

x, y dependent, z is free

no pivots!

x, y can be written in terms of z.

Turn pivots $\longrightarrow$ leading 1's then clear out columns?

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \boxed{-2} & -1 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{2r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in RREF.

leading 1's.

$$\begin{aligned} x + \frac{-3}{2} z &= 0 \\ y + \frac{-1}{4} z &= 0 \end{aligned} \longrightarrow \begin{aligned} x &= \frac{3}{2} z \\ y &= \frac{1}{4} z \end{aligned}$$

The solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/2\, z \\ 1/4\, z \\ 1\, z \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/4 \\ 1 \end{pmatrix} z$$

are all the solutions

z is called free because it can be anything in the solution.

General comments: By hand, RREF is best.

RREF can handle all matrices, square / non square, singular / nonsingular matrices.

Why $PA = LU$? For the computer, this is way easier. (Also useful w/in math...)

Ex $-2x + y + 4w = -1$ # eq'ns < # variable

$x + 2z + 3w = 4$ inf. number of solutions

$$\left[\begin{array}{cccc:c} -2 & 1 & 0 & 4 & -1 \\ \boxed{1} & 0 & 2 & 3 & 4 \end{array}\right] \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 0 & 2 & 3 & 4 \\ -2 & 1 & 0 & 4 & -1 \end{bmatrix}$$

$$\xrightarrow{2r_1+r_2} \begin{array}{cccccc} x & y & z & w & & \\ \end{array}$$

$$\begin{array}{c} \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 4 & 10 & 7 \end{array}\right]$$

$$\begin{array}{cccc} \text{dep} & \text{dep} & \text{free} & \text{free} \end{array}$$

x, y in terms of z, w

$$x + 2z + 3w = 4$$
$$y + 4z + 10w = 7$$

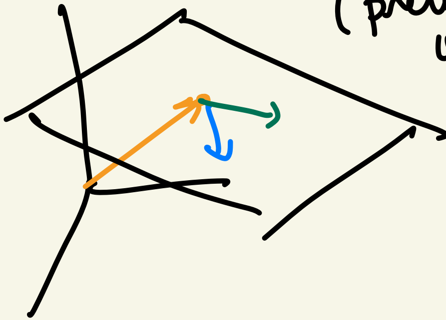
$$\left.\begin{aligned} x &= -2z - 3w + 4 \\ y &= -4z - 10w + 7 \end{aligned}\right\}$$

So the system has the sol'n

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z - 3w + 4 \\ -4z - 10w + 7 \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -3 \\ -10 \\ 0 \\ 1 \end{pmatrix} w + \begin{pmatrix} 4 \\ 7 \\ 0 \\ 0 \end{pmatrix}$$

z, w vary freely.

z, w makes the solution set 2D



$\begin{pmatrix} -2 \\ -4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -6 \\ 0 \\ 1 \end{pmatrix}$ are the "axes" of the solution set.

How do we understand these shapes in an organized way?

<u>Vector Spaces</u>.

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} A = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix}$$

$$\cancel{P} \quad A \qquad\qquad L \qquad\qquad U$$

rearranges y by P^{-1} first

$Ax = b$

$$\Longrightarrow \quad \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \underline{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$L \vec{c} = \vec{b}$$

$a = 0$

$-a + b = -1 \quad b = -1$

$2a + 3b + c = 2 \quad c = 5$

$$c = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix}$$

$Ux = c$

$$\begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad x = \begin{pmatrix} -41/2 \\ -1/2 \\ -5 \end{pmatrix}$$

$$-x + y + 4z = 0 \qquad -x - \tfrac{1}{2} - 20 = 0 \qquad -x = \tfrac{41}{2}$$

$$x = -\tfrac{41}{2}$$

$$2y = -1 \qquad y = -\tfrac{1}{2}$$

$$-z = 5 \qquad z = -5$$

$$A = P^{-1}LU$$

$$\underbrace{\boxed{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1}}}_{\text{rearrange } x, y, z} \underbrace{\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}}_{\begin{pmatrix} x = -41/2 \\ y = -1/2 \\ z = -5 \end{pmatrix}}$$

$1 \to 3$
$2 \to 1$
$3 \to 2$

$1 \to 2$
$2 \to 3$
$3 \to 1$

$x \to y$
$y \to z$
$z \to x$

$\cancel{x = -5}$
$\cancel{y = -41/2}$
$\cancel{z = -1/2}$

nope! Ignore this slide.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$PA = LU$$

$$LU\vec{x} = \vec{b}$$

$$PA\vec{x} = \vec{b}$$

$$A\vec{x} = P^{-1}\vec{b}$$

$A\vec{x}$ = rearranges b

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$A\vec{x} = \vec{b} \qquad \underline{P^{-1}LU}\vec{x} = \vec{b} \qquad LU\vec{x} = \underline{\vec{P}\,b}$$

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 4 \\ & 2 & 0 \\ & & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

now do LU method.

$\cancel{\underbrace{P^{-1}(LU\vec{x}) =} \vec{b}}$ let $\vec{x}$ $LU\vec{x} = b$

$\cancel{P^{-1}b = b}$

Permute $\vec{b}$ by P. then solve LU method

① ②

$$L\overbrace{Ux}^{\vec{c}} = \vec{b}$$

$L\vec{c} = \vec{b}$ solve for $\vec{c}$

$Ux = \vec{c}$ then solve for $\vec{x}$.