To go further, we need a bit better understanding of vectors and their properties. We need vector spaces!

Linear Algebra happens in vector spaces.

Def A real vector space V is a set of objects (vectors) such that you can add vectors $\vec{v}+\vec{w}$, and you can scale by real numbers $c\vec{v}$. And $+$ and $\cdot$ have to satisfy the following properties.

Def A real vector space V is a set of objects (vectors) such that you can add vectors $\vec{v}+\vec{w}$, and you can scale by real numbers $c\vec{v}$. And $+$ and $\cdot$ have to satisfy the following properties.

scalars = all real numbers

1) $\vec{v}+\vec{w} = \vec{w}+\vec{v}$

2) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$

3) There exist a vector $\vec{0}$ such that $\vec{0}+\vec{v} = \vec{v}+\vec{0} = \vec{v}$

4) For $v \in V$, there exist a $-v$ such that $\vec{v}+(\vec{-v}) = \vec{0}$

$-\vec{v}+(\vec{v}) = \vec{0}$

5) $(c+d)\vec{v} = c\vec{v} + d\vec{v}$

5') $c(\vec{v}+\vec{w}) = c\vec{v} + c\vec{w}$

6) $c(d\vec{v}) = (cd)\vec{v}$

7) $1\vec{v} = \vec{v}$ $\quad 1 \in \mathbb{R}$

$\mathbb{R}$ = real numbers

Ex $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n)\}$ is a vector space.

$+, \cdot$ are defined like in lin alg or multi

7 properties are satisfied.

Ex $C^0[a,b]$ = set of all continuous functions w/ inputs in $[a,b]$ and outputs in $\mathbb{R}$

$$[a,b] = \{a \le x \le b\}$$

This is a vector space!

$C^0[0,2\pi]$, objects in here are $\cos(x)$, $\sin(x)$, x^2+x+1, e^x

$\cos(x) + \sin(x)$ is just another function $(\vec{v} + \vec{w})$

$\frac{1}{2}(\cos(x))$ is like scalar multiplication

Functions can be vectors!

$\cos(x)$ is a scalar if x is a scalar

" $x \longleftrightarrow \cos(x)$ for all possible inputs x " is the vector

Def of $+, \cdot$ on $C^0[a,b]$ $f(x)$

$\underbrace{(f+g)}_{\text{sum of vectors (functions)}}(x)$ and $\underbrace{f(x)+g(x)}_{\text{sum of scalars}}$

Def $f+g$ to be $(f+g)(x) = f(x) + g(x)$

Def $(cf)(x) = c \cdot f(x)$

The zero vector $\vec{0}$ is the function $z(x) = 0$

$x \longmapsto 0$

$(-f)(x) = -f(x)$. All distributive properties are satisfied.

Ex Polynomials can also be considered vectors.

Let P = set of all polynomials in the variable "x" and coefficients in $\mathbb{R}$. This is a vector space!

$p(x) + q(x)$ is another polynomial

$(x^2+x+1) + (-x^3+x^2-1) = -x^3+2x^2+x$ is vector addition

$\frac{1}{2}\cdot(x^2+x+1) = \frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{2}$ is scalar mult.

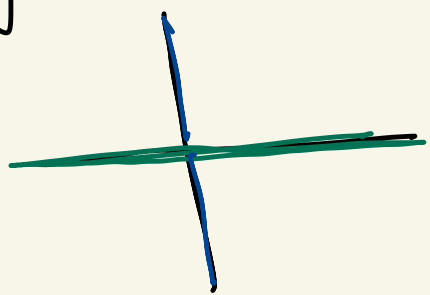
These notions of $+, \cdot$ satisfy the 7 properties so they form a vector space.

Def A subspace W of a vector space V is a subset of V that is a vector space in it's own right.

Ex $V = \mathbb{R}^2$ = xy-plane

$\vec{v} = (a, b)$ $c(a,b) = (ca, cb)$

V



x-axis is a subspace

y-axis is too!

Since the 7 props hold in $\mathbb{R}^2$ they hold on x-axis

W = x-axis $= \{(a,0) \mid a \in \mathbb{R}\}$

$(a,0) + (\bar{a},0) = (a+\bar{a}, 0)$

$c(a,0) = (ca, 0)$, +7 properties!

<u>Non Ex</u> $\{(2,1)\} = W$ is not a subspace.

$$(2,1) + (2,1) \stackrel{?}{=} (4,2) \notin W$$

can't add within W, so not a subspace.

$\vec{0} \notin W$ so W can't be a vector space

<u>Non-Ex</u> $\mathbb{Q} \subseteq \mathbb{R}^1 = \{(a)\}$ $\quad \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \in \mathbb{Q}$

rationals $\frac{a}{b}$ $\quad$ real $\pi, e, \sqrt{2}$

$\frac{1}{2} \cdot \pi \notin \mathbb{Q}$, so you can't scale w/in $\mathbb{Q}$ so it's not a subspace.

Prop / Method of Proof

Let V be a vector space. Suppose $W \subseteq V$ which is a candidate for being a subspace. (use this for 2.2.1, 2.2.2)

Then W is a subspace of V iff

① $\vec{0} \in W$.

② If $\vec{w}, \vec{u} \in W$, then $\vec{w} + \vec{u} \in W$ (add within W)

③ If $c \in \mathbb{R}$, $\vec{w} \in W$, then $c\vec{w} \in W$. (scale w/in W)

Ex
- x-axis $\subseteq \mathbb{R}^2$ satisfies all 3 so it's a subspace.
- $W = (2,1)$ fails all 3 so it's not a subspace.
- $\mathbb{Q}$ only fails ③, so it's not a subspace.

Ex Let $V = \mathbb{R}^3$. $W = \{t(1,2,1) \mid t \in \mathbb{R}\}$.

= all scalar multiples of $(1,2,1)$.

① $\vec{0} \in W$? Well $\vec{0} = (0,0,0) = 0\,(1,2,1) \in W$.

so property 1 is satisfied.

② $\vec{w} = t_1(1,2,1)^*$ In fact $\vec{w} + \vec{u} = t_1(1,2,1) + t_2(1,2,1)$

$\vec{u} = t_2(1,2,1)^*$ distributive property $= \underbrace{(t_1+t_2)}_{\text{scalar}}(1,2,1) \in W$.

W is "closed under addition"

③ Let $c \in \mathbb{R}$, $\vec{w} \in t_1(1,2,1)$. $c\vec{w} = \underbrace{ct_1}_{\text{scalar}}(1,2,1) \in W$.

Since all 3 properties are verified, W is a subspace of $\mathbb{R}^3$.

Let $C^0(\mathbb{R})$ be all cts functions $f : \mathbb{R} \longrightarrow \mathbb{R}$

set of inputs

domain

set of possible outputs

codomain

$\neq$ range of f

$W \subseteq C^0(\mathbb{R})$

$W = \{ f(x) \mid f(0) = 0 \}$

is a subspace, for example.