HW2 due tonight at 11:59 pm.

HW3 will be available later today.

Grades for HW1 should be on canvas | gradescope

Axioms for being a vector space

1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

3) There exist a vector $\vec{0}$ such that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

4) For $v \in V$, there exist a $-v$ such that $\vec{v} + (\vec{-v}) = \vec{0}$

$-\vec{v} + (\vec{v}) = \vec{0}$

5) $(c+d)\vec{v} = c\vec{v} + d\vec{v}$

5') $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

6) $c(d\vec{v}) = (cd)\vec{v}$

7) $1\vec{v} = \vec{v}$ $\quad 1 \in \mathbb{R}$

$(V, +, \cdot)$

$$0\vec{v} = \vec{0}$$

$$-1\vec{v} = -\vec{v}$$

You can prove properties like this from the axioms.

**Prop** Let $V$ be a vector space. Then

a) $\underline{0\vec{v} = \vec{0}}$ *

b) $-1\vec{v} = -\vec{v}$

c) $c\vec{0} = \vec{0}$

d) $c\vec{v} = \vec{0}$, then either $c = 0$ or $\vec{v} = \vec{0}$.

1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

3) There exist a vector $\vec{0}$ such that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

4) For $v \in V$, there exist a $\underline{-v}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$, $-\vec{v} + (\vec{v}) = \vec{0}$

5) $(c+d)\vec{v} = c\vec{v} + d\vec{v}$

5') $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$

6) $c(d\vec{v}) = (cd)\vec{v}$

7) $1\vec{v} = \vec{v}$  $1 \in \mathbb{R}$

**Pf** of a).

$$\boxed{0\vec{v}} \underset{3)}{=} 0\vec{v} + \vec{0} \underset{4)}{=} 0\vec{v} + \underbrace{(0\vec{v} + (\underline{-0\vec{v}}))}$$

$$\underset{2)}{=} (0\vec{v} + 0\vec{v}) + -0\vec{v} \underset{5)}{=} (0+0)\vec{v} + -0\vec{v}$$

$$= 0\vec{v} + -0\vec{v} \underset{4)}{=} \boxed{\vec{0}} \quad \square$$

# Non-Example

Let $V$ = set of angles in radians

$= [0, 2\pi) = \{\theta \mid 0 \le \theta < 2\pi\}$

$\theta^\circ$ is a vector candidate.

$\theta^\circ + 4^\circ$ you can add angles

$\pi + \pi = 2\pi = 0$

$c\,\theta^\circ$ scaling angles $\quad V, +, \cdot$

But it's not a vector space!

$\theta + \theta' = \theta' + \theta$

$\vec{0} = 0$ degrees $= 0$ radians

$-\theta$ = negative vector

1) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓

2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ✓

3) There exist a vector $\vec{0}$ such that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ ✓

4) For $v \in V$, there exist a $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$, $-\vec{v} + (\vec{v}) = \vec{0}$ ✓

5) $(c+d)\vec{v} = c\vec{v} + d\vec{v}$ ✓

5') $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ ✓

6) $c(d\vec{v}) = (cd)\vec{v}$ ✗

7) $1\vec{v} = \vec{v}$ $\quad 1 \in \mathbb{R}$ ✓

$c = \frac{1}{2} \quad d = 2 \quad \vec{v} = \pi = 180^\circ$

$c(d\vec{v}) = \frac{1}{2}(2\pi) = \frac{1}{2}(0) = 0$

$\neq$

$(cd)\vec{v} = \left(\frac{1}{2} \cdot 2\right)(\pi) = 1\pi = \pi$

Property 6 fails, not a V.S

$\frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi = \pi$ $\qquad 270° + 270° = 180°$

Defining "vector addition" and "scalar mult" this way fails ⑥ so

it's not a v.s.

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$\vec{v} + \vec{w}$ , $c\vec{w}$ $\qquad \cancel{\vec{v} \cdot \vec{w}} = ??$

Every "algebraic" thing you can do in a v.s looks like this

$$\vec{w} = c\vec{v}_1 + c\vec{v}_2 + \dots + c\vec{v}_n$$

This is what's called a linear combination.

Ex $V = \mathbb{R}^3$ $\quad \underline{c\begin{pmatrix}1\\2\\1\end{pmatrix} + d\begin{pmatrix}-1\\2\\-1\end{pmatrix} + e\begin{pmatrix}5\\-1\\0\end{pmatrix}}$ can be simplified

Ex $\underbrace{\tfrac{1}{2}}_{c_1}\underbrace{\cos^2(x)}_{\vec{v}_1} + \underbrace{(-3)}_{c_2}\underbrace{\sin(x)}_{\vec{v}_2} + \underbrace{5}_{c_3}\underbrace{(1)}_{\vec{v}_3}$ cannot be simplified

Ex $\underbrace{5}_{c_1}\underbrace{(x^2+x+1)}_{\vec{v}_1} - \underbrace{3}_{c_2}\underbrace{(x^2-1)}_{\vec{v}_2} + \underbrace{\tfrac{5}{2}}_{c_3}\underbrace{(x^3-1)}_{\vec{v}_3}$

$$\underline{c\begin{pmatrix}1\\2\\1\end{pmatrix} + d\begin{pmatrix}-1\\2\\-1\end{pmatrix} + e\begin{pmatrix}5\\-1\\0\end{pmatrix}} = \begin{pmatrix}c-d+5e\\2c+2d-e\\c-d\end{pmatrix}$$

Ex $\quad 1\sin^2(x) + 1\cos^2(x) + (-1)\,1 = 0$

no row reduction in function vector spaces e.g. $C^0(\mathbb{R})$

constant function $f(x) = 1$

zero function

Ex $\quad c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\-1\end{pmatrix} = \vec{0}$

$$\begin{pmatrix}c_1 - c_2\\ c_1 - c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\longrightarrow$ solve for $c_1, c_2$ by row reduction if you wanted to!

Studying linear combinations in different vector spaces can be different. But they have the following in common...

<u>Def</u> Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V$. Then

$$\mathrm{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \left\{ \begin{array}{l} \text{set of all linear combinations} \\ \text{of } \vec{v}_1, \ldots, \vec{v}_n \end{array} \right\}$$

$$= \left\{ c_1\vec{v}_1 + \ldots + c_n\vec{v}_n \;\middle|\; \begin{array}{l} c_1 \ldots c_n \text{ allowed to} \\ \text{vary} \end{array} \right\}$$

This called the <u>span</u> of $\vec{v}_1 \ldots \vec{v}_n$.

**Prop** $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ is a subspace of $V$.

**Pf** Well, $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n$

so $\vec{0} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_n)$

(Plug in $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_n = 0$.)

✓ $0 \in$ span

✓ $u+w \in$ span

✓ $cw \in$ span

Let $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n \in \text{span}(v_1, \ldots, v_n)$

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$$

$$= (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_n + d_n)\vec{v}_n$$

$$\in \text{span}(v_1, \ldots, v_n)$$

Let $\alpha \in \mathbb{R}$,

then $\alpha(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)$

$$= (\alpha c_1)\vec{v}_1 + (\alpha c_2)\vec{v}_2 + \dots + (\alpha c_n)\vec{v}_n$$

$$\in \operatorname{span}(v_1, \dots, v_n) \qquad \square$$

$(x+iy) + (u+iv) = (xu) + i(vy)$ (Possibility)

$W = \{(x,y,z) \mid x+y+z+1 = 0\} \subseteq \boxed{\mathbb{R}^3}$ known v.s.

all 7 already true on W.

Not a subspace?

$(x,y,z) = (0,0,0) = \vec{0}.$

$\vec{0} \notin W$

Need to check $+, \cdot$ make sense on W.

Subspace: vector inside another one

$(0,0,-1) \in W$. But $\frac{1}{2}(0,0,-1) \notin W$.

counterexample to property 3

$\boxed{\text{For all}}$ $c$, $\vec{w} \in W$, $c\vec{w} \in W$ also.

$$U = \begin{pmatrix} u_{11} & & * \\ & u_{22} & \\ & & \ddots \\ & & & u_{nn} \end{pmatrix}$$

anything

$u_{ii} \neq 0.$

zero entries

Contradiction proof

Assume $\vec{x} \neq \vec{0}$.

$$\left.\begin{aligned} u_{11}x_1 + \cdots + u_{1n}x_n &= 0 \\ u_{22}x_2 + \cdots + u_{2n}x_n &= 0 \\ &\vdots \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= 0 \\ u_{nn}x_n &= 0 \end{aligned}\right\} \Longrightarrow x_n = \frac{0}{u_{nn}} = 0$$

$u_{nn} \neq 0.$

$$\underset{\neq 0}{u_{n-1,n-1}}\, x_{n-1} + 0 = 0$$

$$x_{n-1} = 0.$$

Continuing by back substitution

$$\vec{x} = 0\,!$$

If $u_{nn} = 0$ then $\underline{0 = 0}$

$$\begin{bmatrix} * & * & * & \\ & * & * & \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \infty \text{ number of solutions}$$

$Ux = 0$ has nontrivial sol'n end

$\Downarrow \quad \Updownarrow \quad \Uparrow$

These statements imply each other.

$U_{ii} = 0$ for some $i$. start

if and only if = biconditional

$U_{ii} \neq 0 \Rightarrow \vec{x} = 0.$

$\vec{x} \neq 0 \Leftarrow \Rightarrow U_{ii} = 0$ for some $i$.

nontrivial solution

$$\begin{pmatrix} u_{11} & & & & \\ & u_{22} & & & \\ & & 0 & * & * \\ & & & u_{44} & * \\ & & & & u_{55} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$\cancel{*}$

$0$ you can cancel these by row reduction

$$\begin{pmatrix} u_{11} & * & * & * & * \\ & u_{22} & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ & & & u_{44} & 0 \\ & & & & u_{55} \end{pmatrix} \longrightarrow \begin{pmatrix} u_{11} & \cdots & & & \\ & u_{22} & \cdots & & \\ & & 0 & u_{44} & 0 \\ & & & & u_{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $x$ nontrivial