Last time:

$$\text{span}(v_1 \dots v_k) = \left\{ \begin{matrix}\text{all linear combinations} \\ v_1 \dots v_k\end{matrix} \right\}$$

Let $v_1 \dots v_k \in V$ where V is a vector space.

then $W = \text{span}(v_1 \dots v_k)$ is a subspace of V.

Questions:

How many vectors do you need to write

$V = \text{span}(v_1 \dots v_n)$? (Tomorrow)

Can every subspace be written as $W = \text{span}(v_1 \dots v_k)$?

of $\mathbb{R}^n$

2 2.1 $W = \{ (x,y,z) \mid x - y + 4z = 0 \}$

Can this set be written as a span?

$= \{ a\vec{v}_1 + b\vec{v}_2 \}$?

$$W = \left\{ (x,y,z) \;\middle|\; \underbrace{\begin{array}{ccc} x & y & z \\ \hline \boxed{1} & -1 & 4 \end{array}}_{\text{matrix to row reduce}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$$

Already in RREF!

x is dependent, y, z free

$x - y + 4z = 0 \implies x = y - 4z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\begin{pmatrix} y-4z \\ y \\ z \end{pmatrix}} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -4z \\ 0 \\ z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}\underbrace{y}_{\alpha} + \underbrace{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}}\underbrace{z}_{\beta}$$

$$W = \underline{\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

why?
$$\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v} \in \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad v = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} a + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} b$$

$$= \left(\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) a + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} b$$

* distributive property from v.s. axioms

$$= \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} a + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} (a+b)$$

$$= \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2 \in \text{Span}\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Span}\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \text{span}\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Q: Given a vector $\vec{w}$ and a span

$W = \text{span}(\vec{v}_1, \dots \vec{v}_k)$, how do we decide

if $w \in \text{span}(v_1, \dots, v_k)$?

Can we find coefficients such that

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{w} \, ?$$

Ex $V = C^0(\mathbb{R})$ = vector space of continuous functions

$f : \mathbb{R} \longrightarrow \mathbb{R}$

inputs - domain

potential outputs - codomain

$W = \text{span}(\cos^2(x), \sin^2(x))$.

Is $f(x) = 1 \in \text{span}(\cos^2(x), \sin^2(x))$?

$(1)\cos^2(x) + (1)\sin^2(x) = 1$ so yes

$1 \in \text{span}(\cos^2(x), \sin^2(x))$!

Is $1 \in \text{Span}(\cos(x), \cos(2x), \cos(3x), \cos(4x))$?

Harder question in $V = C^0(\mathbb{R})$.

But in $\boxed{\mathbb{R}^n}$, asking $w \in \text{span}\underbrace{(v_1 \ldots v_k)}$ is computable by row reduction!

write them as columns

Claim: Let $A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}$ $n \times k$

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

Then $\underbrace{A\vec{c}}_{\text{matrix product}} = \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k}_{\text{general element of } \mathrm{span}(v_1 \ldots v_k)}$.

Pf: If we write $\vec{v}_i = (v_{ji})_{1 \le j \le n}$

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & & v_{1k} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & & v_{nk} \end{pmatrix}$$

$$A\vec{c} = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ \vdots & \vdots & & \vdots \\ v_{n1} & v_{n2} & & v_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} c_1 v_{11} + \cdots + c_k v_{1k} \\ \vdots \\ c_1 v_{n1} + \cdots + c_k v_{nk} \end{pmatrix}$$

$$n \times \cancel{k} \qquad \cancel{k} \times 1$$

$$= c_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{n1} \end{pmatrix} + \cdots + c_k \begin{pmatrix} v_{1k} \\ \vdots \\ v_{nk} \end{pmatrix} = c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k \,!$$

Ex Let's work in $\mathbb{R}^3$. $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ & $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Is $\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}\right)$?

GOAL: Potentially find $\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

$$c_1\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}2\\3\\-1\end{pmatrix} = {*}\ \underbrace{\begin{pmatrix}-1&2\\0&3\\1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}3\\3\\-2\end{pmatrix}}_{\text{We just need to solve this!}}\ ?$$

$$\left(\begin{array}{cc|c}-1&2&3\\0&3&3\\1&-1&-2\end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc}1&0&-1\\0&1&1\\0&0&0\end{array}\right)$$

∞ number of solutions

Tell us what $c_1\ c_2$ are!

There does exist c_1, c_2 such that

$$\begin{pmatrix}-1\\0\\1\end{pmatrix}c_1 + \begin{pmatrix}2\\3\\-1\end{pmatrix}c_2 = \begin{pmatrix}3\\3\\-2\end{pmatrix}.$$

Yes $\begin{pmatrix}3\\3\\-2\end{pmatrix} \in \text{Span}\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-1\end{pmatrix}\right)$!

$$\underline{-1\begin{pmatrix}-1\\0\\1\end{pmatrix} + 1\begin{pmatrix}2\\3\\-1\end{pmatrix}} = \begin{pmatrix}3\\3\\-2\end{pmatrix}$$

Def: Let V be a vector space and $w, v_1, \ldots v_k \in V$.

We say $v_1 \ldots v_k$ are linearly dependent if

$\exists$ $c_1 \ldots c_k \neq 0$ such that

$$\underbrace{c_1 v_1 + \cdots + c_k v_k}_{} = \vec{0}.$$

there exists

at least one is non-zero.

$W \in \text{span}(v_1 \ldots v_k) \implies \{w, v_1, v_2, \ldots, v_k\}$ is linearly dependent.

w depends on $v_1 \ldots v_k$.

Def : We say $v_1 \dots v_k$ are linearly independent

if $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$

then $c_1 = c_2 = \dots = c_k = 0$.

(The only possible way to make a linear relation between independent vectors is w/ $0v_1 + 0v_2 + \dots + 0v_k$ i.e. no relation at all.)

$$\left(w \notin \text{span}(v_1 \dots v_k) \implies w \text{ is independent from } v_1 \dots v_k \right)$$

Often times you'll be asked, "are $v_1 \dots v_k$ independent or dependent"?

Let's work in $\mathbb{R}^3$. $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Is $\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} \in \text{Span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right)$?

3

$$\Updownarrow$$

Are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$ linearly independent or linearly dependent?

<u>More general phrasing</u>

Thm (Summary of discoveries from today)

Let $v_1 \ldots v_k \in \mathbb{R}^n$ $\quad A = \begin{pmatrix} \vec{v}_1 & \ldots & \vec{v}_k \end{pmatrix}$

i) $v_1 \ldots v_k$ are dependent $\iff$ $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = 0$

$\iff$ $A\vec{c} = 0$ has nontrivial solution

ii) $v_1 \ldots v_k$ are independent $\iff$ only $c_1 = c_2 = \ldots = 0$ satisfies

$c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = 0$

$\iff$ $A\vec{c} = 0$ only has trivial solution.

(A has enough pivots!)

iii) $w \in \mathrm{Span}(v_1 \dots v_k) \iff c_1\vec{v}_1 + \dots c_k\vec{v}_k = \vec{w}$

$\iff \quad A\vec{c} = \vec{w}$ has non trivial sol'n.

$$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 0 & 3 & 3 \\ 1 & -1 & -2 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

free

1 free variable $\longleftrightarrow$ 1 dimensions worth of relationships between columns

$$(-1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

$$-1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 3 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -1 & 2 & 3 \\ 0 & 3 & 3 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} -t \\ t \\ -t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{only need } t}$$

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$$

More than 2 vectors in $\mathbb{R}^2$ are dependent!