Linearly Independent vectors $v_1 \dots v_k$

$\Longleftrightarrow$ $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$ means that $c_1 = c_2 = \dots = 0$

__Def__ Let V be a vector space. We say $w_1 \dots w_m$ __span V__ when all vectors $v \in V$ are in the span of $w_1 \dots w_m$

i.e. $V = \operatorname{span}(w_1 \dots w_m)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

__Ex__ $w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\boxed{w_3 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$ dependent on $w_1 w_2$

$\operatorname{span}(w_1 w_2 w_3) \subseteq \mathbb{R}^2$. In fact

$\operatorname{span}(w_1 w_2 w_3) = \mathbb{R}^2$. They __span $\mathbb{R}^2$__.

Every vector in $\mathbb{R}^2$ is a linear comb of these 3 vectors.

<u>Def</u> Let V be a vector space. We say a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$ forms a <u>basis</u> of V if

1) $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

2) $\vec{v}_1, \ldots, \vec{v}_n$ span V.

2) Says that all vectors $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ are linear combinations of $v_1 \ldots v_n$

1) But since $v_1 \ldots v_n$ are independent, no redundant vectors.

You can think of a basis as a <u>maximally</u> independent set. in that if I consider $\{v_1, \ldots v_n, w\}$ is no longer independent. $w = c_1v_1 + \ldots + c_nv_n \Rightarrow$ dependent!

Def Let V be a vector space. We say a set of vectors $\vec{v}_1, \dots, \vec{v}_n$ forms a <u>basis</u> of V if

✓ 1) $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent

✓ 2) $\vec{v}_1, \dots, \vec{v}_n$ span V.

Ex $V = \mathbb{R}^3$

$$\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$ forms a basis of $\mathbb{R}^3$.

1) Suppose $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{0}$.

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{0}$$

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \vec{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

So $\vec{e}_1\ \vec{e}_2\ \vec{e}_3$ are independent! ✓

2) Claim $\text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = \mathbb{R}^3$

Given a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ ✓ $\square$.

$e_1 e_2 e_3$ span $\mathbb{R}^3$.

Since $e_1 e_2 e_3$ are independent and span, they form a basis.

Ex let $V = \mathbb{R}^n$

Def let $\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} spot $\in \mathbb{R}^n$

This is called the i^{th} standard basis vector of $\mathbb{R}^n$.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

standard basis of $\mathbb{R}^3$

* In fact $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ forms a basis of $\mathbb{R}^n$.

Pf $c_1\vec{e}_1 + \cdots + c_n\vec{e}_n = 0 \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $c_1 = c_2 = \cdots = c_n = 0$

independent!

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$ so they span. $\Rightarrow$ Basis!

Ex $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a basis of $\mathbb{R}^2$.

✓ 1) Independent $\longrightarrow$ $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ $\rightsquigarrow$ $c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

✓ 2) Span

$$\boxed{\underbrace{\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{pmatrix}$$

independent vectors = # of leading 1's.

2 independent vectors

$\longrightarrow$ Independent!

Given $\begin{pmatrix} x \\ y \end{pmatrix}$, can we write it as

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 2 \end{pmatrix} ?$$

This is how to show $v_1 v_2$ span $\mathbb{R}^2$

$c_1\ c_2$ in terms of x, y.

invertible!

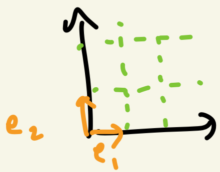
$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

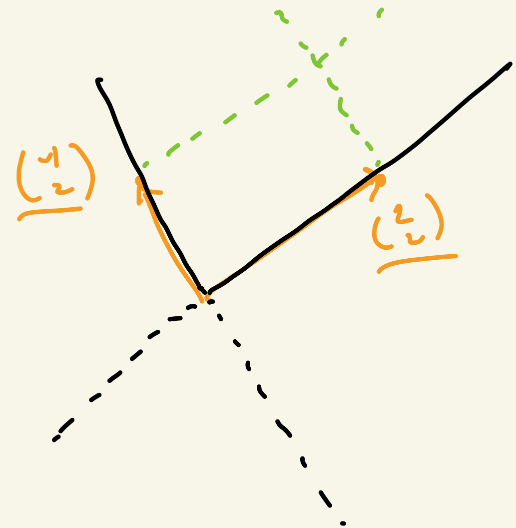
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-6}\begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$c_1 = -\frac{1}{6}(2x) + \frac{1}{6}(2y)$ $\quad c_2 = \frac{-1}{6}(-2x) + \frac{1}{6}y$

So $\begin{pmatrix} x \\ y \end{pmatrix} \in \text{span}\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right)$ so $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ span $\mathbb{R}^2$.

$\Longrightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ form a basis!

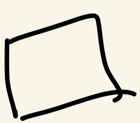




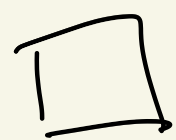
e_1 e_2

choice of

x, y axis



Basis $\longleftrightarrow$ axes



<u>Ex</u> But there's no finite basis for function vector spaces

$C^0[a,b]$ = continuous functions on $[a,b]$

$\vec{f}_1 = e^x \quad \vec{f}_2 = e^{2x} \quad \vec{f}_3 = e^{3x}, \quad f_4 = e^{4x}, \ \ldots$

are all independent!

Infinite independent vectors $\Longrightarrow$ no finite basis.

$\vec{p}_0 = 1, \ \vec{p}_1 = x, \quad \vec{p}_2 = x^2, \ \vec{p}_3 = x^3,$ etc are all independent.

Finite combinations $\Longrightarrow$ polynomial

Infinite combinations $\Longrightarrow$ $f(x) = \underline{\sum a_n x^n}$ infinite series

We'll stick to finite for now.

$C^0[a,b] \supseteq \text{Span}(\cos^2(x), \sin^2(x), 1)$ is "finite" subspace of $C^0[a,b]$.

Claim: $\text{Span}(\cos^2(x), \sin^2(x), 1)$ has basis $\cos^2(x), \sin^2(x)$.

Why?

Ex P = vector space of all polynomials $\quad 1, x, x^2, x^3, \ldots$

$P^{(n)}$ = vector space of polynomials up to degree n.

$\vec{p} = a_0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n \text{ "=" } \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$

$1, x, x^2, \ldots, x^n$ is a basis of $P^{(n)}$. IOU.

Prop: Let $\vec{v}_1 \dots \vec{v}_n$ be a $\boxed{\text{basis}}$. Then any vector $\vec{v} \in V$ is a <u>unique</u> linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

Note: Span tells $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$; at least 1 l.c.

There's <u>only 1</u> !

($\text{WTS } c_i = d_i$)

GOAL

Pf: Assume $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

$$\vec{v} = d_1\vec{v}_1 + \dots + d_n\vec{v}_n.$$

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = d_1\vec{v}_1 + \dots + d_n\vec{v}_n$$

$$\underbrace{(c_1 - d_1)}_{0}\vec{v}_1 + \underbrace{(c_2 - d_2)}_{0}\vec{v}_2 + \dots + \underbrace{(c_n - d_n)}_{0}\vec{v}_n = 0 \quad *$$

Since $v_1 \dots v_n \longrightarrow$ $\boxed{\text{1) independent}}$

2) span

$$(c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \ldots + (c_n - d_n)\vec{v}_n = 0 \quad *$$

This is a linear combination of independent vectors $= 0$

So $c_i - d_i = 0 \implies c_i = d_i$ $\square$.

Thm Let V be a vector space, w/ bases $\{v_1 \ldots v_n\}$ (n vectors)

and $\{w_1 \ldots w_m\}$ (m vectors). Then $n = m$.

(All bases have the same size!)

So size of a basis is an inherent feature of V!

Def We say dimension of V is the size of one of it's bases. $(\dim(V) = n.)$

Continue next time