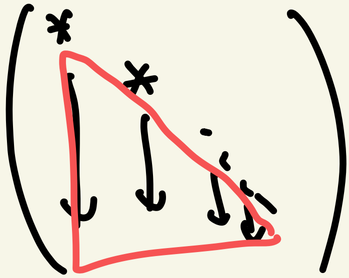
Goal: Take a regular matrix A

not including row swapping



$$r_j' = c r_i + r_j \qquad j > i.$$

$$A = LU.$$

row operation e $\rightsquigarrow$ Elementary matrix E_e

$$e_1 \, e_2 \, e_3 \dots e_n : A \longrightarrow U$$

$$U = E_{e_n} \cdots E_{e_1} A$$

Since A was regular,

the E_{ℓ_i} are lower triangular,

$$U = \tilde{L} A \,, \quad \tilde{L} = E_{\ell_n} \cdots E_{\ell_1}$$

"Compute $(\tilde{L})^{-1}$"

$$r_j' = c r_i + r_j \qquad \begin{array}{cc} & \begin{array}{cccc} & i & & \end{array} \\ \begin{array}{c} \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & c & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

Backwards row operation is:

$$r_j' - c r_i = r_j$$

$$\boxed{r_j = -c r_i + r_j'}$$

If I do the operations

$$r_j' = c r_i + r_j \quad *$$

then

$$r_j' = -c r_i + r_j \quad *$$

I get back the original A!

$$\overbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -c & & 1 \end{pmatrix}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & c & & 1 \end{pmatrix} A = A$$

$* L_i$ $\quad * E_i$

Every regular matrix is nonsingular but every nonsingular matrix is regular.

If $\underline{U} = E_m E_{m-1} \ldots \ldots E_1 \underline{A}$

A is regular so E_i are lower Δ

Let L_i be the elementary matrix which reverses E_i.

L_i is also lower Δ.

$$\underbrace{L_1 L_2 \ldots \ldots L_m}_{L} U = (L_1 L_2 \ldots L_m E_m \ldots \ldots E_1) A$$

undoes row reduction

row reduction

$$(L_1 \ldots L_m) U = I_n A$$

$$\boxed{LU = A}$$

☆ !

So the <u>LU decomposition</u>

of a regular matrix A

is $A = LU$

where U is the

Upper Δ row reduction of A

and L is the product

of all of the reverse

elementary matrices.

$5r_2 + r_3$ $\qquad$ 3×3

$$\longrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & 5 & 1 \end{pmatrix} = E$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} = L$$

Example: Find the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 1 & -3 \\ \boxed{2} & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \xleftarrow{2r_1+r_2}$$

$$\xrightarrow{-2r_1+r_2} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ \boxed{3} & 0 & -2 \end{pmatrix} \xleftarrow{3r_1+r_3}$$

$$\xrightarrow{-3r_1+r_3} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & \boxed{-3} & 7 \end{pmatrix} \xleftarrow{r_2+r_3}$$

$$\xleftarrow{-r_3}$$

$$\xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix} = U$$

Then

$$A = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$$

Claim:

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{pmatrix}$$

The LU decomposition is

$$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -3 \\ & -3 & 8 \\ & & -1 \end{pmatrix}$$

$$A \qquad = \qquad L \qquad U$$

§ 1.4 Permuted LU decomposition

~~Regular~~ + row swapping

+ $cr_i + r_j \quad j > i$

$$\boxed{\begin{pmatrix} 0 & 5 & 3 \\ 1 & 1 & 1 \\ -2 & 0 & 5 \end{pmatrix}} \xrightarrow{\text{swap } 1,2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ \boxed{-2} & 0 & 5 \end{pmatrix} *$$

$$\xrightarrow{2r_1 + r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & \boxed{2} & 7 \end{pmatrix}$$

To get back here we need a permutation matrix

$$\xrightarrow{-\frac{2}{5}r_2 + r_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 29/5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ -2 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -2 & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 29/5 \end{pmatrix}$$

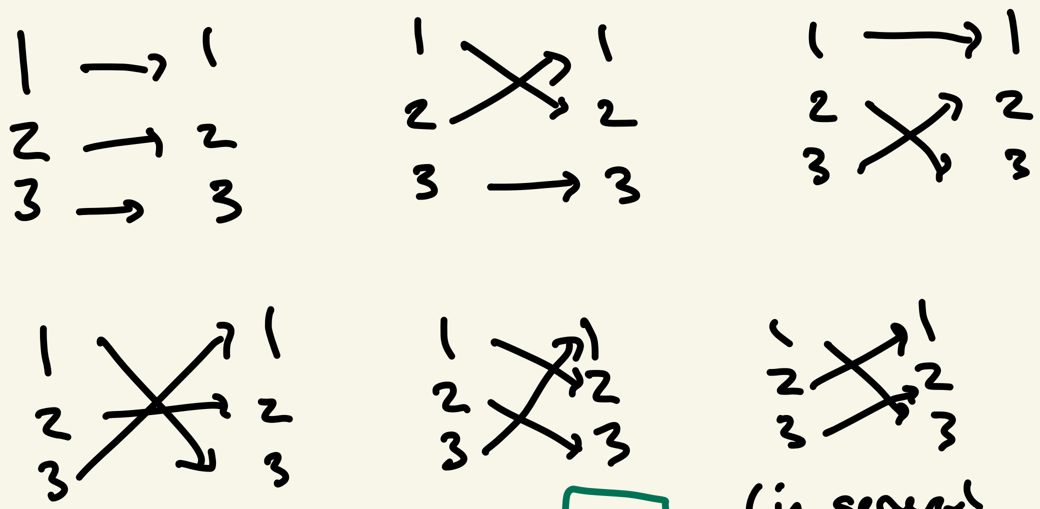
Permutations

A permutation is a way to rearrange n objects.

Ex $n = 3$

$S_3 = \{\text{all permutations on 3 things}\}$

$\quad = \{\text{all ways to rearrange } 1, 2, 3\}$



$6 = \boxed{3!}$ (in general you get $n!$)

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \underline{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}} \quad \underline{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}$$

Let's say we have a column vector

<u>Ex</u>

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \xrightarrow[1,2]{} \begin{pmatrix} b \\ a \\ c \end{pmatrix} \xrightarrow[2,3]{} \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \xrightarrow[1,2]{} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

$$\xrightarrow[2,3]{} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

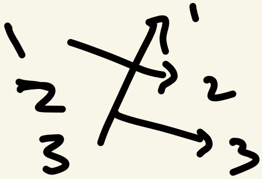
So $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is <u>permutation</u>

<u>matrix</u> associated to

$$\begin{tikzcd} 1 \arrow[dr] & 1 \\ 2 \arrow[dr] & 2 \\ 3 \arrow[uur] & 3 \end{tikzcd}$$

This happens in general!

Every way to rearrange $\{1, \ldots, n\}$ has a corresponding $n \times n$ **permutation matrix**



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{columns}]{\text{permute}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

To make corresponding permutation matrix, permute the columns of ID matrix.

If we add row swapping

$$\underline{PA} = LU$$

P is a permutation matrix

if I do all the permuting of rows of A at the beginning of row reduction,

then PA should be regular, so

$$PA = LU.$$

This is the <u>permuted LU decomposition</u>.

Pivots

$$\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ \boxed{1} & 1 & 3 \end{pmatrix} \xrightarrow{-r_1+r_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & \boxed{-1} & 2 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_3} \begin{pmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{-1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 2 & 2 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \xrightarrow[1,2]{\text{row swap}} \boxed{\xrightarrow[L]{2r_1+r_2}} \xrightarrow[2,3]{\text{row swap}} U$$

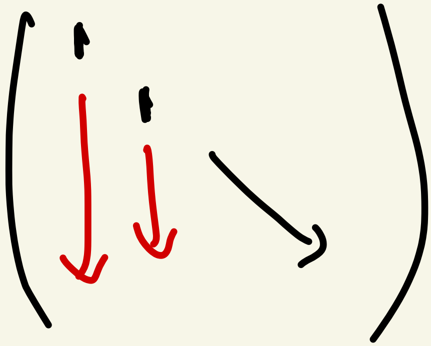
$$A \xrightarrow[1,2]{\text{row swap}} \xrightarrow[2,3]{\text{row swap}} \xrightarrow{2r_1+r_3} U$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \underbrace{\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\underbrace{\begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix}\underbrace{\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} A}_{1}}_{2}}_{3}$$

$$\underline{\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}} A = LU$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{pmatrix} U$$



$$EP = P\underline{E}'$$