§1.8/1.9

§1.9 Determinants

Given a matrix $A \in M_{n\times n}(\mathbb{R})$
or $M_{n\times n}(\mathbb{C})$

$\det A$ is a scalar quantity,

- if $\det A \neq 0$
  then A is nonsingular
  or invertible.

- A is not invertible
  if and only if
  $\det A = 0$.

This implies that **most** matrices are inv.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2 \neq 0$$

in fact $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1}$

$$= \frac{1}{\boxed{-2}}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

<u>Ex</u> $A = \begin{pmatrix} 5 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ top row expansion

$$\det A = 5\det\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} - 1\det\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$+ 2\det\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$$

$$= 5\cdot 3 - 1\cdot(-1) + 2(-5)$$

$$= 15 + 1 - 10 = 6 \neq 0$$

so A^{-1} exists.

Thm Given any square matrix A, there exists a unique scalar $\det A$ such that

- doing $cr_i + r_j$ to A preserves the determinant
- swapping 2 rows changes the sign of the determinant
- the operation $r_i' = cr_i$, the determinant gets scaled by c.

uses permutation comp.

- Given any upper Δ matrix U, $\det U = \prod_{i=1}^{n} u_{ii}$

This thm is a formula for how to compute the det by row reduction. (Axiomatic Approach)

$$\begin{matrix} 5 & 1 & 2 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \end{matrix} \quad \det 6$$

$$\xrightarrow{\text{swap } 1,3} \quad \begin{matrix} 1 & 2 & 1 \\ -1 & 3 & 0 \\ 5 & 1 & 2 \end{matrix} \quad \det = -6$$

$-r_1+r_2$

$$\xrightarrow{r_1+r_2} \quad \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 5 & 1 & 2 \end{matrix} \quad \det = -6$$

$5r_1+r_3$

$$\xrightarrow{-5r_1+r_3} \quad \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & -9 & -3 \end{matrix} \quad \det = -6$$

$-\frac{9}{5}r_2+r_3$

$$\xrightarrow{\frac{9}{5}r_2+r_3} \quad \begin{matrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -\frac{6}{5} \end{matrix} \quad = U$$

$$\det U = (1)(5)\left(-\frac{6}{5}\right) = -6$$

$$\det A = 6$$

A matrix is nonsing. if

$A \xrightarrow{\text{row. red.}} U$ w/ nonzero entries on diagonal

if $U_{ii} \neq 0 \quad \forall i$

then $\det U \neq 0 \implies \det A \neq 0.$

$$\det A = 2\begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ -1 & 1 \end{vmatrix} = 6$$

Permutation for the Determinant

Given an $n \times n$ matrix A

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i \underbrace{a_{\sigma(1),1} \cdots a_{\sigma(n),n}}$$

permutations on n objects

± 1

$$\det A = \sum_{\sigma \in S_n} \underbrace{\operatorname{sgn}(\sigma)}_{\pm 1} \prod a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

$n = 3$

Remember there are 6 permutations

$\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix}$ $+1$

$\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix}$ -1

$\begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix}$ -1

$\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}$ -1

$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$ $(-1)^2 = +1$

$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$ $+1$

The $\operatorname{sgn}(\sigma) = (-1)^{\text{\# of switches it takes to do perm.}}$

$$\det A = +1\, a_{11} a_{22} a_{33}$$
$$- a_{21} a_{12} a_{33}$$
$$- a_{31} a_{22} a_{13}$$
$$- a_{11} a_{32} a_{23}$$
$$+ a_{21} a_{32} a_{13}$$
$$+ a_{31} a_{12} a_{23}$$

$n=2$

$12 \quad {+1}$ $\qquad 12 \quad {-1}$

$12 \qquad\quad 21$

$$\det \begin{pmatrix} \boxed{a} & \boxed{b} \\ \boxed{c} & \boxed{d} \end{pmatrix} = ad - bc$$

This is probably the most "theoretically useful" formula.

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§1.8 General Gaussian Elimination

If you have m eq'ns and n variables the matrix A in $Ax=b$ is not square. (if $m \neq n$)

Row operation still applies

Given a system $A\tilde{x} = \tilde{b}$
w/ $A \in M_{m\times n}(\mathbb{R})$

To solve the system row reduce to

$$\begin{pmatrix} \boxed{1} & * & \cdots & * & 0 & & & 0 & & & 0 & & & \\ 0 & 0 & & 0 & \boxed{1} & * & * & 0 & & & 0 & & & \\ \vdots & \vdots & & \vdots & 0 & 0 & 0 & \boxed{1} & * & * & 0 & & & \\ & & & & & \vdots & & 0 & 0 & 0 & \boxed{1} & * & * & * \\ & & & & & & & & \vdots & & & \vdots & & \\ 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & & & 0 & & \end{pmatrix}$$

This is reduced row echelon form.

It makes reading the answer to a system as easy as possible.

If all nonzero rows have first nonzero entry 1, called leading 1's, and every other entry in a column of a leading 1 is zero.

All nonzero rows are at top of matrix.

Ex

$$\left(\begin{array}{cccc|c} 0 & 0 & 3 & 5 & 2 \\ 1 & -2 & -1 & 2 & 1 \\ -1 & 2 & 1 & 1 & 0 \end{array}\right)$$

3 eq'ns

4 unknowns

LD 1

$$\xrightarrow{\text{swap } 1,2} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 5 & 2 \\ \boxed{-1} & 2 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r_1 + r_3} \left(\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 2 & 1 \\ 0 & \boxed{0}\,? & \boxed{3} & 5 & 2 \\ 0 & 0 & 0 & \boxed{3} & 1 \end{array}\right)$$

3 is a pivot (future leading 1)

Already in echelon

Back substitute

$$\xrightarrow[\frac{1}{3} r_3]{\frac{1}{3} r_2} \left(\begin{array}{cccc|c} \boxed{1} & -2 & -1 & 2 & 1 \\ 0 & 0 & \boxed{1} & \frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \boxed{1} & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow{-\frac{5}{3} r_3 + r_2} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow{-2r_3 + r_1} \left(\begin{array}{cccc|c} 1 & -2 & \boxed{-1} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

$$r_2 + r_1 \quad \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} \boxed{1} & -2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & \boxed{1} & 0 & \frac{1}{9} \\ 0 & 0 & 0 & \boxed{1} & \frac{1}{3} \end{array}\right) \end{array}$$

$$x - 2y = \frac{4}{3}$$

$$z = \frac{1}{9}$$

$$w = \frac{1}{3}$$

Any column w/out leading 1 corresponds to free variable!

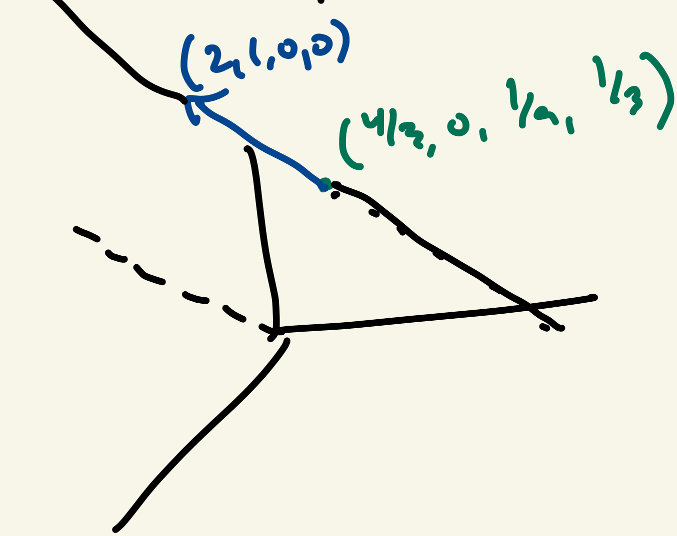
y is free. Let $y = s$

$x = 2s + \frac{4}{3}$ $\qquad$ $z = 0s + \frac{1}{9}$

$y = 1s + 0$ $\qquad$ $w = 0s + \frac{1}{3}$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 4/3 \\ 0 \\ 1/a \\ 1/3 \end{pmatrix}.$$

This a general sol'n to the system!



To study these kinds of sol'n in a lot of different contexts, people started introducing <u>vector spaces</u>.