Reminders

- HW3 due 6/16
- Exam 1 is in class Friday 6/19

Chapter 2

§2.1 Vector Spaces

These objects provide a way to understand linear algebra in one nice framework.

- Linear systems

$$\downarrow$$

matrix multiplication problem

$$\downarrow$$

$$\begin{bmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{bmatrix}$$

regular matrices unique solution

$$\begin{bmatrix} 1 & * & * & 0 & & \\ & & & 1 & * & * \\ & & & & & \end{bmatrix}$$

In general, solutions aren't unique.

What about this matrix makes the solution not unique? What kind of solutions can you get?

Def A **real vector space**

is a set V equipped with

- vector addition

  if $v, w \in V$, $v + w \in V$

- scalar multiplication

  given $c \in \mathbb{R}$, $v \in V$

  $cv \in V$.

These operation satisfy the following axioms.

(a) $v + w = w + v$

(b) $u + (v + w) = (u + v) + w$

(c) There exists a vector $\vec{0}$ s.t.

$$v + \vec{0} = \vec{0} + v = v$$

(d) Given any $v \in V$, there's a vector $-v$

$$v + (-v) = -v + v = \vec{0}.$$

(e) $(c+d)v = cv + dv$

$c(v+w) = cv + cw$

(f) $c(dv) = (cd)v$

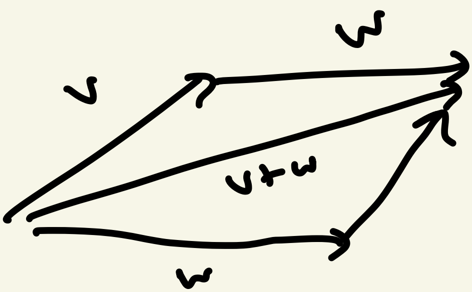
(g) $1v = v$, $(1 \in \mathbb{R})$

Any set V w/ $+, \cdot$ such that these axioms are satisfied is a vector space over $\mathbb{R}$.

In the past,

vectors $\longrightarrow$ $(3, -1, 5)$

vectors don't need to look like this!



Being a vector doesn't mean it is of the form

$$(a, b, c, \ldots, z),$$

any set w/ those properties is a vector space!

<u>Ex:</u> Let $C^0(\mathbb{R})$ denote the set of continuous functions $\mathbb{R} \to \mathbb{R}$

$$\left(\lim_{x \to a} f(x) = f(a) \right)$$

This set $C^0(\mathbb{R})$ is a real vector space!

In this case, <u>functions</u> are the <u>vectors</u>.

So to show $C^0(\mathbb{R})$ is a v.s,
we need to define $+, \cdot$
and show they satisfy the axioms.

Let $f, g \in C^0(\mathbb{R})$.

Define $f+g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$(f+g)(x) = f(x) + g(x)$$

Given a $c \in \mathbb{R}$, define a
function cf by

$$(cf)(x) = c \cdot f(x)$$

Lets show that $+, \cdot$ satisfy
the axioms!

(a) $f+g = g+f$

Pf

$(f+g)(x) = f(x)+g(x) = g(x)+f(x)$

$= (g+f)(x)$

Thus $f+g = g+f$.

(Remember! $f+g$ is also continuous!)

(b) $f, g, h \in C^0(\mathbb{R})$, $f+(g+h) = (f+g)+h$

$\big(f+(g+h)\big)(x)$

$= f(x) + (g+h)(x)$

$= f(x) + \big(g(x)+h(x)\big)$

$= \big(f(x)+g(x)\big) + h(x)$ assoc. of $\mathbb{R}$

$= (f+g)(x) + h(x)$

$= \big((f+g)+h\big)(x)$.

(c) $C^0(\mathbb{R})$ has a 0 element.

Define $\vec{0}(x) = 0.$

In fact

$$(f+\vec{0})(x) = f(x) + \vec{0}(x)$$
$$= f(x) + 0$$
$$= f(x)$$

$f + \vec{0} = f.$ ($\vec{0}$ is cts.)

Similarly, $\vec{0} + f = f.$ $\blacksquare$

$0 \neq \vec{0}$

scalar!

function!

$c \in \mathbb{R} \quad \neq \quad f(x) = c$

Scalars are not the same as constant functions!

(d) Need to define $-f$. Given a function $f(x)$, define

$$(-f)(x) = -f(x).$$

(Note: This is also cts.)

Need to show $f + (-f) = \vec{0}$

Pf

$$\big(f + (-f)\big)(x) = f(x) + (-f(x))$$

$$= f(x) - f(x) = 0.$$

Thus $f + (-f) = \vec{0}$ as functions.

(e) $(c+d)f = cf + df$

Pf

$$\big((c+d)f\big)(x) = (c+d)f(x)$$

distr of $\mathbb{R}$

$$= cf(x) + df(x)$$

$$= (cf)(x) + (df)(x).$$

$$= (cf + df)(x)$$

$c(f+g) = cf + cg.$

Pf $\quad (c(f+g))(x) = c \cdot (f+g)(x)$

$$= c(f(x) + g(x))$$

$$= c \cdot f(x) + c \cdot g(x)$$

$$= (cf)(x) + (cg)(x)$$

$$= (cf + cg)(x)$$

(f) $\quad c(df) = (cd)f$

$$(c(df))(x) = c \cdot (df)(x)$$

$$= c \cdot d \cdot f(x)$$

$$= (c \cdot d) \cdot f(x)$$

$$= ((cd)f)(x)$$

(g) $\quad (1f)(x) = 1 \cdot f(x) = f(x)$

So $C^0(\mathbb{R})$ is a vector space!

$\underline{\text{Ex}}$ $\mathbb{R}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R}\}$

This is a vector space!

$(a_1, \ldots, a_n) + (b_1, \ldots, b_n)$

$= (a_1 + b_1, \ldots, a_n + b_n).$

$c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n).$

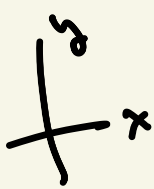
$\mathbb{R}^n, +, \cdot$ satisfy the 7 axioms!

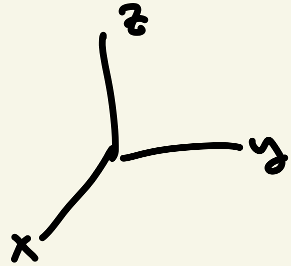
So its a vector space.

$\vec{0} = (0, \ldots, 0)$ $\quad -v$

$= (-a_1, \ldots, -a_n)$

$n=2$, xy-plane $\qquad$ $n=3$ xyz-space





In this course $\mathbb{R}^n$ will be the $n \times 1$ matrices.

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$ instead.

Conflating $(a_1, \ldots, a_n)$ w/

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

or $(a_1 \ \ldots \ a_n)^T$.

Prop For any vector space, the following hold:

(i) $0\vec{v} = \vec{0}$.

(ii) $(-1)v = -v$ The axioms tell us this exists only abstractly.

(iii) $c\vec{0} = \vec{0}$.

(iv) If $cv = 0$, then $c = 0$ or $\vec{v} = \vec{0}$.

Pf $0\vec{v} \overset{(c)}{=} 0\vec{v} + \vec{0} \overset{(d)}{=} 0\vec{v} + (0\vec{v} - 0\vec{v})$

$\overset{(b)}{=} (0\vec{v} + 0\vec{v}) + (-0\vec{v})$

$\overset{(e)}{=} (0+0)\vec{v} + -(0\vec{v})$

$= 0\vec{v} + -0\vec{v} \overset{(c)}{=} \vec{0}$

(ii) Pf

$$
\begin{aligned}
(-1)v &= (-1)v + v + (-v) \\
&= ((-1)v + v) + (-v) \\
&= ((-1)v + 1v) + (-v) \\
&= (-1+1)v + (-v) \\
&= 0v + (-v) \\
&= \vec{0} + (-v) \\
&= -\vec{v}.
\end{aligned}
$$

Further Examples

$M_{m \times n}(\mathbb{R})$ is a real vector space.

$$\underline{A+B} \in M_{m \times n}(\mathbb{R})$$

$$cA \in M_{m \times n}(\mathbb{R})$$

The 7 axioms are proved in §1.2.

As a vector space,

$M_{m \times n}(\mathbb{R})$ is the same as $\mathbb{R}^{mn}$.

$$\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{pmatrix} \rightsquigarrow \begin{pmatrix} a_{11} \\ \vdots \\ \vdots \\ a_{mn} \end{pmatrix}$$

But they're different in that $M_{m \times n}$ has AB. So keep them different.

Ex $C^1(\mathbb{R})$ is the vector space of differentiable functions.

$$C^1(\mathbb{R}) \subset C^0(\mathbb{R}).$$

Ex $P = \{\text{polynomials in } x\}$ is a vector space.

$$(3x^2 + 2) + (5x^3 + (-2)x^2)$$

$$= 5x^3 + x^2 + 2.$$

$$3(2x^2 + 1) = 6x^2 + 3.$$