Non-Examples of Vector Spaces

① The integers $\mathbb{Z}$

$$= \{ \dots -3, -2, -1, 0, 1, 2, 3, \dots \}$$

are not a vector space!

Let $n, m \in \mathbb{Z}$, $n+m \in \mathbb{Z}$

We have "vector addition".

But there is no scalar multiplication.

$c = \frac{1}{2}$, $n \in \mathbb{Z}$. Unless n is even, then $\frac{1}{2} n \notin \mathbb{Z}$.

(Maybe $n = 5$, $\frac{1}{2} \cdot 5 \notin \mathbb{Z}$)

② Let $I = [0,1]$. This is not a vector space either.

$v = \frac{2}{3}$, then $2v = \frac{2}{3} + \frac{2}{3}$

$= \frac{4}{3} \notin [0,1]$.

Don't have vector addition.

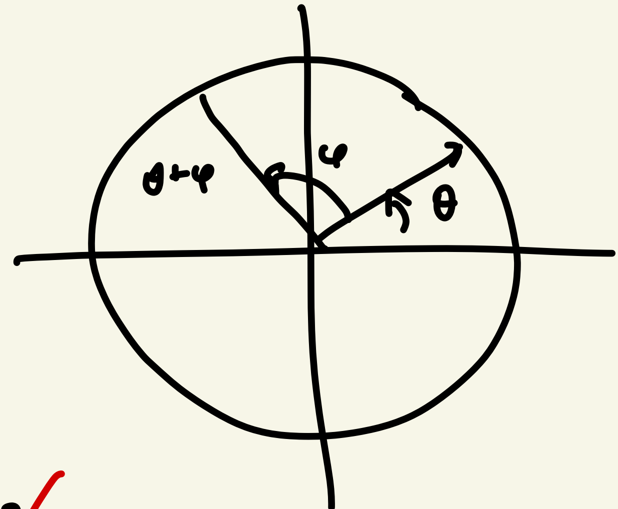
In general $I = [a,b]$ or $I = (a,b)$ aren't vector spaces.

③ Let $C = [0, 2\pi)$, the set of angles around a circle.

Each of our angles θ is a "vector".

To add vectors, just add angles.

$\theta + \varphi$



$\pi + \pi = \cancel{2\pi}$

$= 0.$

Scalar mult. is as expected

$\frac{1}{2} \cdot \pi = \frac{\pi}{2}$, for example.

$+, \cdot$ are well-defined. but C is not a vector space.

a) $\theta + \varphi = \varphi + \theta$ ✓

b) $(\theta + \varphi) + \psi = \theta + (\varphi + \psi)$ ✓

c) 0° is the 0 element ✓

d) $-\theta$ are well defined ✓

e) distributive prop holds ✓

f) Assoc. of scalar mult. doesn't hold ✗

$$\left(\frac{1}{2} \cdot 2\right) \cdot \pi = \left(\frac{1}{2} \cdot 2\right)\pi = \pi$$

$$\neq$$

$$\frac{1}{2} \cdot (2 \cdot \pi) = \frac{1}{2} \cdot 0 = 0$$

Since this axiom failed, C is not a vector space!

§ 2.2. Subspaces

Def Let V be a vector space. A subspace W of V is a subset $W \subseteq V$ that is a vector space in it's own right, addition and scalar mult. inherited from V.

$+ : V \times V \longrightarrow V$ $\qquad$ $\cdot : \mathbb{R} \times V \longrightarrow V$

$+(v,w) = v+w$ $\qquad$ $\cdot(c,v) = cv$

$+$ on W is the restriction of the $+$ function on V to W.

$$\left(+_W = +_V\big|_W\right) \qquad \left((\cdot_W) = (\cdot_V)\big|_W\right)$$

A subspace is a vector space inside of another vector space.

<u>Def</u> A subspace W of V is a $*$ subset $W \subseteq V$ s.t. ($\forall$ = for all)

$\forall v, w \in W$, $v + w \in W$ ($W \neq \emptyset$)

$\forall v \in W$ $\forall c \in \mathbb{R}$, $cv \in W$.

W is <u>closed</u> under addition and scalar mult.

These two definitions are equivalent!

In practice, to show a set W is a subspace, you only need closure under $+$ and $\cdot$.

Pf Def 1 $\Rightarrow$ Def 2

If W is a vector space, then

$+ : W \times W \longrightarrow W$.

In particular, the $v+w \in W$ $\forall v, w \in W$.

Same for scalar mult.

Def 2 $\Rightarrow$ Def 1 $(W \subseteq V)$

If W is closed under add. and scalar mult. then it's a vector space.

a) $v+w = w+v$ ✓

b) $v+(w+u) = (v+w)+u$ ✓

c) 0 element $(0 \cdot v = 0 \in W)$ ✓

(Note: If $0 \notin W$, then W is not a subspace.)

d) If $w \in W$, then $-w = (-1)w \in W$.

So W has additive inverses.

e) ✓

f) ✓

g) $1v = v$ ✓

So therefore W is a vector space

and Def 2 $\Rightarrow$ Def 1.

Since $v+w \in W$ $\forall v, w$

$+ : W \times W \longrightarrow W$ is well defined.

Same for $\cdot : \mathbb{R} \times W \to W$

<u>Ex</u> Let V be a vector space.

$W = \{0\}$ is a subspace and

$V \subseteq V$ is a subspace of itself.

If $W = \{0\}$.

$0 + 0 = 0 \in W$ ✓

$c \cdot 0 = 0 \in W$ ✓

So W is a subspace.

If $V \subseteq V$,

$v + w \in V$ ✓

$cv \in V$ ✓

These are called the <u>trivial</u> subspaces.

If $v = 0$ to start, these are the same.

could be anything

Ex Let $V = \mathbb{R}^3$. Let $\boxed{\begin{array}{l}(a,b,c)\\ = (1,2,3)\end{array}}$

Let

$$W = \{(x,y,z) \mid (1,2,3)\cdot(x,y,z) = 0\} *$$

$$= \{(x,y,z) \mid (1,2,3) \perp (x,y,z)\} *$$

$$= \{(x,y,z) \mid x + 2y + 3z = 0\} *$$

Let $v, w \in W$. Let

$v = (v_1 v_2 v_3)$, $w = (w_1 w_2 w_3)$

Now $\underline{v + w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

$$(1,2,3)\cdot v + w = 1(v_1 + w_1) + 2(v_2 + w_2) + 3(v_3 + w_3)$$

$= v_1 + w_1 + 2v_2 + w_2 + 3v_3 + 3w_3$

$= \underbrace{(v_1 + 2v_2 + 3v_3)}_{v \in W} + \underbrace{(w_1 + 2w_2 + 3w_3)}_{w \in W}$

$= 0 + 0 = 0$

$v + w \in W$ as well.

let $c \in \mathbb{R}$,

$cv = (cv_1, cv_2, cv_3)$

$cv_1 + 2(cv_2) + 3(cv_3)$

$= c(v_1 + 2v_2 + 3v_3)$

$= c \cdot 0 = 0$

$cv \in W$. $\quad 0 \in W$, so $W \neq \emptyset$.

W is a subspace!

Planes in $\mathbb{R}^3$ are subspaces.

Def Let $\mathcal{F}(\mathbb{R})$ be <u>all</u> functions $\mathbb{R} \to \mathbb{R}$. (possibly discont.)

This is a v.s.

Claim: $C^0(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R})$.

($f+g$ is also cts)

(cf is also cts)

Let $C^1(\mathbb{R})$ denote all functions s.t. f' and f' is cts.

Claim: $C^1(\mathbb{R}) \subseteq C^0(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$

subspace subspace

$C^1(\mathbb{R})$

$$(f+g)' = f' + g' \checkmark$$

Addition is closed

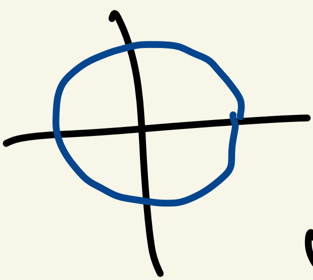
$$(cf)' = cf' \checkmark$$

Scalar mult. is closed

Non - Examples

① Let $S^1 \subseteq \mathbb{R}^2$

$$S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$$



Not a subspace

$0 \notin S^1 \quad 0^2 + 0^2 \neq 1.$

$(1,0) \in S^1$, but $(1,0) + (1,0) = (2,0) \notin S^1.$

$2(1,0) \notin S^1.$

(2) Half Plane.

$H \subseteq \mathbb{R}^2$,

$$H = \{(x,y) \mid y \geq 0\}$$

This is not a subspace.

$$(x,y) + (x',y') \in H$$

$$y + y' \geq 0$$

$$-\frac{1}{2}(x,y) = \left(-\frac{1}{2}x, -\frac{1}{2}y\right)$$

$$-\frac{1}{2}y \not\geq 0.$$

So not closed under scaler mult.

Not a subspace.

③ $V = M_{n \times n}(\mathbb{R}) \; (\sim \mathbb{R}^{n^2})$

$W = \{\text{invertible matrices}\}$

$:= GL_n(\mathbb{R})$

(general linear)

is not a subspace.

0 is not an inv. matrix.

If A is invertible, then so is

$-A$.

But $A + (-A) = 0 \notin GL_n(\mathbb{R})$.

not closed under add. ✗

$(cA)^{-1} = \frac{1}{c} A^{-1}$ (unless $c = 0$)

✗

Practice Problems

① Let P the vector space of polynomials in 1 var.

$W = \{ p(x) \; p(5) = 0 \}$.

If $p, q \in W$

$(p+q)(5) = p(5) + q(5)$

$= 0 + 0 = 0$

$p + q \in W$ ✓

$(cp)(5) = cp(5) = c \cdot 0 = 0.$

$cp \in W.$ ✓

$W \neq \phi \quad (x-5) \in W.$

(2) $W = \{ z = x^2 + y^2 \}$ not a subspace.

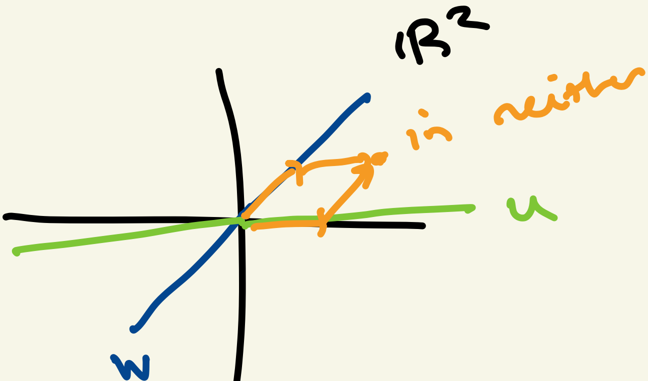
$(1,1,2) \in W$

but $(-1,-1,-2) \notin W$.

you can scale by -1.

Not closed under scalar mult.

(3) let $U, W \subseteq V$ be subspace of V. when is $U \cup W$ a subspace?



In general $U \cup W$ is not a subspace.

Claim: $U \cup W$ is a subspace iff $U \subseteq W$ or $W \subseteq U$.

Pf: Assume $U \cup W$ is a subspace.

In particular, $\forall u \in U, \forall w \in W$

$u + w \in U \cup W$. (in U or W)

Fix, u, w, assume WLOG (without loss of generality) that $u + w \in U$.

$u + w = u' \in U$

But $u' - u = w \in U$

Therefore $W \subseteq U$.

Rain Check! I'll post it to canvas.

§ 2.3 Span and Linear Independence

Def Let V be a v.s. Let $v_1 \ldots v_k \in V$ and $c_1 \ldots c_k \in \mathbb{R}$.

Then a linear combination of the vectors $v_1 \ldots v_k$ is a term of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k \in V.$$

Def Let $v_1 \ldots v_k \in V$. Define the span of $v_1 \ldots v_k$ as the set $W = \{$all linear combinations of $v_1 \ldots v_k\}$

$$= \{ c_1 v_1 + \ldots + c_k v_k \mid c_i \in \mathbb{R} \}$$

Def Let $W \subseteq V$ be a subspace of V.

Then a spanning set for W is a set of vectors $v_1 \ldots v_k$ such that $W = \text{span}(v_1 \ldots v_k)$

$\text{span}(v_1 \ldots v_k) = \{$ set of linear combinations of $v_1 \ldots v_k \}$

Prop For all sets of vectors $v_1 \ldots v_k$

$\text{span}(v_1 \ldots v_k)$ is a subspace.

Pf Let $\vec{v} = c_1 v_1 + \ldots + c_k v_k$

$\vec{w} = d_1 v_1 + \ldots + d_k v_k$

$$v + w = c_1 v_1 + \ldots + c_k v_k + d_1 v_1 + \ldots + d_k v_k$$

$$= (c_1 + d_1) v_1 + \ldots + (c_k + d_k) v_k \in \text{span}(v_1 \ldots v_k) \checkmark$$

$a(c_1 v_1 + \dots + c_k v_k) \qquad a \in \mathbb{R}$

$= (ac_1)v_1 + \dots + (ac_k)v_k$

$\in \mathrm{span}(v_1 \dots v_k)$ ✓

So $\mathrm{span}(v_1, \dots, v_k)$ is a subspace of V.

<u>Ex</u> $V = \mathbb{R}^3$

$\mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$

$= \mathbb{R}^3$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{c}{3}\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} = ?$

$*$

Claim: If $A = (v_1 \cdots v_k)$

and row reduce to

$U = (u_1 \dots u_k)$

then $\mathrm{span}(v_1 \dots v_k)$ "=" $\mathrm{span}(u_1 \dots u_k)$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ \boxed{-2} & -3 & 1 \\ \boxed{1} & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{span}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\neq \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right)$$

$$= \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

The third is redundant, but spans are the same size not the same exact subspace.

If $v_k \in \text{Span}(v_1 \ldots v_{k-1})$

then

$$\text{span}(v_1 \ldots v_k) = \text{span}(v_1 \ldots v_{k-1})$$

$$-2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{Span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right)$$

$$= \text{Span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right)$$

This example was really about linearly dependent vs. linearly independent

Def The vectors $v_1 \dots v_k$ are called **linearly dependent** if $\exists\ (c_1, \dots, c_k) \neq 0$

s.t. $c_1 v_1 + \dots + c_k v_k = 0$.

(if $\exists$ non-trivial linear combination of $v_1 \dots v_k$ which is 0.)

Ex

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ are linearly dependent.

$$(-2)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + (1)\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0.$$

Def Vectors $v_1 \dots v_k$ are linearly independent if

$$c_1 v_1 + \dots + c_k v_k = 0$$

$$\Longrightarrow \underline{c_i = 0} \quad \forall i.$$

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ are linearly independent in $\mathbb{R}^3$.

Pf: If

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 0 \\ \underline{c_3 = 0} \end{array}$$

Thus they're independent.

Given, $v_1 \dots v_k \in \mathbb{R}^n$

$c_1 v_1 + c_k v_k = 0$ is a

linear system of equations.

$$v_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix} \dots \quad v_k = \begin{pmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{pmatrix}$$

$$c_1 v_1 + \dots + c_k v_k$$

$$= c_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{n1} \end{pmatrix} + \dots + c_k \begin{pmatrix} v_{1k} \\ \vdots \\ v_{nk} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 v_{11} \\ \vdots \\ c_1 v_{n1} \end{pmatrix} + \dots + \begin{pmatrix} c_k v_{1k} \\ \vdots \\ v_{nk} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 v_{11} + \dots + c_k v_{1k} \\ \vdots \\ c_1 v_1 + \dots + c_k v_{nk} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} & \cdots & v_{1k} \\ \vdots & & \vdots \\ v_{n1} & & v_{nk} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$c_1 v_1 + \dots + c_k v_k = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \vec{0}$$

$$\Updownarrow$$

$$A\vec{x} = 0$$

This is a <u>homogeneous</u> system of equations ($\vec{b} = \vec{0}$).

Thm Let $v_1 \ldots v_k \in \mathbb{R}^n$,

Let $A = (v_1 \ldots v_k)$.

- The vectors $v_1 \ldots v_k$ are dependent iff $A\vec{c} = 0$ has a nonzero sol'n.

- The vectors are linearly independent iff the system $A\vec{c} = \vec{0}$ only has the $c = 0$ sol'n.

Pf Write out $A\vec{c} = 0$ like we just did.

Ex Determine whether $\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}2\\-3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$
$\quad\quad\quad v_1 \quad\quad v_2 \quad\quad v_3$

are dependent or independent.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

nothing happens!!

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad \text{free} \end{array}$$

$$\xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This has a non-unique sol'n and the vectors are dependent!

$$-2v_1 + v_2 = v_3$$

$C^0(\mathbb{R})$

$1, x, x^2$ are independent functions

If $\underline{c_1 + c_2 x + c_3 x^2} = \underline{0}$

(as functions !!) (quadratics only have 2 roots.)

then $c_1 = 0$ $c_2 = 0$ $c_3 = 0$.

Objects: vector spaces

2.1.13

Maps: Linear Transformations

$$\begin{tikzcd}
 & Z \arrow[dl, bend right, "\pi_1"'] \arrow[d, "\exists!"] \arrow[dr, bend left, "\pi_2"] & \\
V & V \times W \arrow[l, "p_1"] \arrow[r, "p_2"'] & W
\end{tikzcd}$$

$V \times W$ is the "best vector space w/ projection maps.

$p_1(v, w) = v$

$p_2(v, w) = w$

$\mathbb{R}^{n+m}$ has same property above as $\mathbb{R}^n \times \mathbb{R}^m$.

$\mathcal{F}(S) = \{\text{all functions } f : S \to \mathbb{R}\}$

Let $f, g : \boxed{S} \to \mathbb{R} \in \mathcal{F}(S)$

$$(f+g)(s) = \underbrace{f(s) + g(s)}_{\text{addition in } \mathbb{R}}$$

$$= g(s) + f(s)$$

$$= (g+f)(s)$$

2.1.1

$$(a+bi)(c+di)$$

$$= (ac - bd) + (bc + ad)i$$

This mult. has nothing to do w/ $\mathbb{C}$ being a vector space over $\mathbb{R}$.

$\mathbb{C}$ $+$ $(a+bi) + (c+di)$

$\cdot$ $\alpha \cdot (a+bi) = (\alpha a + \alpha b i)$

$\vec{v}\,\vec{w}$ never happens.

As vector spaces

$a + bi \longleftrightarrow (a,b)$

$(a+bi)+(c+di)$ $\quad$ $(a,b)+(c,d)$

$(a+c)+i(b+d)$ $\quad$ $(a+c,\ b+d)$

$\mathbb{R}^2 \simeq \mathbb{C}$

isomorphic

2.1.13

$$\mathbb{R} \times \mathbb{R} = \{(x, y)\} = \mathbb{R}^2$$

$$\mathbb{R}^n \times \mathbb{R}^m$$

$$= \{((x_1 \dots x_m), (y_1 \dots y_m))\}$$

$$= \{(x_1 \dots x_n, y_1 \dots y_m)\}$$

$$= \mathbb{R}^{n+m}$$

$$\mathbb{R}^2 \times \mathbb{R}^5 = ((x_1, x_2), (y_1\ y_2\ y_3\ y_4\ y_5))$$

$$= (x_1, x_2\ y_1\ y_2\ y_3\ y_4\ y_5)$$

$$= \mathbb{R}^7$$

Let (v_1, v_2, v_3) satisfy

$$x - y + 4z = 0$$

and (w_1, w_2, w_3) satisfy

$$x - y + 4z = 0.$$

Need to show that

$$v + w = (\underline{v_1 + w_1},\ \underline{v_2 + w_2},\ \underline{v_3 + w_3})$$

satisfies $x - y + 4z = 0.$

$$(a, b, c) = (1, -1, 4)$$