③ If U, W are subspaces of a vector space V, then $U \cup W$ is a subspace iff $U \subseteq W$ or $W \subseteq U$.

Pf Assume $U \cup W$ is a subspace.

Assume for contradiction

$U \not\subseteq W$ and $W \not\subseteq U$.

Then there exist a $u \in U$ s.t. $u \notin W$. Similarly let $w \in W$ s.t. $w \notin U$.

$u + w \in U \cup W$ (since it's a subspace)

either $u + w \in U$ or $u + w \in W$.

If $u + w \in U$ let $u' = u + w$,

$$w = u' - u \in U. \text{ (since } U \text{ is a subspace)}$$

But $w \notin U$. Contradiction

If $u + w \in W$, let $w' = u + w$

$$u = w' - w \in W$$

But $u \notin W$. So contradiction!

Thus either $U \subseteq W$ or $W \subseteq U$.

Basically, $U \cup W$ is almost never a subspace!

If $U \subseteq W$ or $W \subseteq U$ then

$U \cup W = U$ or W, in either case, it's a subspace.

$\square$

§ 2.3 Basis and dimension

Def Let V be a vector space.

A basis of V is a set $\beta = \{v_1, \ldots, v_n\}$ such that β is linearly independent and a spanning set.

Ex Standard basis of $\mathbb{R}^n$.

Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} row $\in \mathbb{R}^n$.

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ etc.

$I = (e_1\ e_2\ \cdots\ e_n)$.

Def We call e_i the i^{th} standard basis vector of $\mathbb{R}^n$.

Def The set $\{e_1, e_2, \ldots, e_n\}$ is called the standard basis of $\mathbb{R}^n$.

The set $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

Pf Independent

Let $c_1 e_1 + \ldots + c_n e_n = 0$.

$$\Rightarrow c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + c_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This eq'n tells us that

$$c_1 = 0 \;,\; c_2 = 0, \ldots, c_n = 0.$$

Thus $\{e_1, e_2, \ldots, e_n\}$ is independent!

<u>Span</u>

$$\text{Span}(e_1, \ldots, e_n) = \mathbb{R}^n.$$

By def $\text{span}(e_1, \ldots e_n) \subseteq \mathbb{R}^n$.

let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

Then $v_1 e_1 + v_2 e_2 + \ldots + v_n e_n$

$$= \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \vec{v}.$$

Thus $\vec{v} \in \text{Span}(e_1, \ldots, e_n)$

and $\text{Span}(e_1 \ldots e_n) = \mathbb{R}^n$ $\square$

A vector space has more than 1 basis!

- $\mathbb{R}^3$
  - $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$
  - $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ ← come back to this

(Since there's more than 1 basis, some mathematicians like to talk about vector spaces w/out bases entirely!

A "coordinate-free" description.)

Thm Suppose V has bases $\ast$
$\beta = \{v_1 \ldots v_n\}$ and $\beta' = \{w_1 \ldots w_m\}$.

Then $n = m$.

Any two bases have the same size!

Def Let V be a vector space
w/ basis $\beta = \{v_1, \ldots v_n\}$. Then
the dimension of V is the size

of β. $\dim V = n$.

So computing the dim. of a vector space
consists of finding a basis and
computing the size of that basis.

Ex $\dim \mathbb{R}^n = n$. (The standard basis has n vectors.)

Note: If V has a basis of finite size, i.e $\dim V < \infty$

we say V is **finite dimensional**.

Ex $C^0(\mathbb{R})$ is infinite dimensional.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + \cdots$$

Is this a linear combination of

$1, x, x^2, x^3, \cdots$?

Pf of *

Lemma Suppose $v_1 \dots v_n$ span V.
Let $\{w_1 \dots w_k\} \subseteq V$. Then this set of w_i is dependent if $k > n$.

Pf of lemma

Since $v_1 \dots v_n$ span V.

Then $w_j = \sum_{i=1}^{n} a_{ij} \boxed{v_i}$

could be anything

Then $c_1 w_1 + \dots + c_k w_k = 0$ has a nontrivial sol'n $c_1 \dots c_k$

iff

$$c_1\left(\sum a_{i1} v_i\right) + \dots + c_k\left(\sum_i a_{ik} v_i\right) = 0$$

$$\sum_{j=1}^{k} \sum_{i=1}^{n} c_j a_{ij} v_i = 0$$ has a non-trivial sol'n.

$\Longleftrightarrow$

$$\sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{k} c_j a_{ij} \right)}\boxed{v_i} = 0$$ has a non-trivial sol'n.

$\Longleftrightarrow$ The system of eq'ns

$\in \mathbb{R}$

① $$\sum_{j=1}^{k} \boxed{c_j} \boxed{a_{1j}} = 0$$

unknowns.

② $$\sum_{j=1}^{k} c_j a_{2j} = 0$$

$\vdots$

ⓝ $$\sum_{j=1}^{k} c_j a_{nj} = 0$$

has a non-trivial sol'n.

But this system has k unknowns and n eq'ns w/ $k > n$.

Let $A = \begin{pmatrix} a_{11} & & \\ & \cdots & a_{nk} \end{pmatrix} \in M_{n \times k}(\mathbb{R})$.

Then $A\vec{c} = \vec{0}$ is the matrix form of the linear system.

Since $k > n$, there are more columns than pivots or leading 1's.

Therefore A has a free column, and thus a non-trivial solution.

$\Longrightarrow c_1 w_1 + \dots + c_k w_k = 0$ has a non-trivial solution.

Thus, $\{w_1 \dots w_k\}$ is dependent. $\square$

Pf of thm.

Let $\beta = \{v_1 \ldots v_n\}$, $\beta' = \{w_1 \ldots w_m\}$.

Suppose that $m > n$.

Then since β is a basis, these vectors $\{v_1 \ldots v_n\}$ span.

The lemma implies that $\{w_1 \ldots w_m\}$ is dependent since $m > n$.

But β' was a basis, this is a contradiction.

Then $m \leq n$.

But argument works in reverse!

If $n > m$, we get a contradiction

So $n \leq m$. Both together, imply $n = m$. $\square$

Thm Suppose V is a v.s w/ $\dim V = n$

(a) If $k > n$, then $\{w_1 \ldots w_k\}$ is dependent.

(b) If $k < n$, then $\operatorname{span}\{w_1 \ldots w_k\} \subsetneq V$.

* (c) A set of n vectors $\{w_1 \ldots w_n\}$ is a basis iff it spans.

* (d) A set of n vectors $\{w_1, \ldots w_n\}$ is a basis iff it's an independent set.

same proof + row reduction

Ex Any set of n independent vectors in $\mathbb{R}^n$ is a basis.

Q: How to prove $\{v_1 \ldots v_n\} \subseteq \mathbb{R}^n$ is a basis?

A: Put $\{v_1 \ldots v_n\}$ in the columns of a matrix. Then row reduce.

If you get n pivots, then $\{v_1 \ldots v_n\}$ are independent $\Rightarrow$ they're a basis.