Reminder: Exam 1 6/19 (tomorrow!)

- Check email for details
- 5 problems
- 50 min + 10 min to upload

  10:10am - 11:10am

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§2.5 Fundamental Subspaces of Matrices

Let $A \in M_{m \times n}(\mathbb{R})$
m rows
n columns

<u>Def</u> let <u>kernel</u> of A, $\ker(A)$, be the set

$$\ker(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.$$

Def $\ker(A) = \{ x \in \mathbb{R}^n \mid A\vec{x} = 0 \}$

= all of the solutions to the homogeneous system associated to A

$$A \cdot \vec{x} \in \mathbb{R}^m$$

$$m \times \cancel{n} \cdot \cancel{n} \times 1 \qquad m \times 1$$

But $x \in \mathbb{R}^n \implies \ker(A) \subseteq \mathbb{R}^n$

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad T_A(\vec{x}) = A\vec{x}$$

$$\vec{x} \longmapsto A\vec{x}$$

The kernel $\subseteq \mathbb{R}^n$ is the set of all vectors $x \in \mathbb{R}^n$ that get mapped to 0.

$$\ker(A) = T_A^{-1}(0).$$

<u>Def</u> let img(A) is the set

$$\operatorname{img}(A) = \left\{ v \in \mathbb{R}^m \;\middle|\; v = Ac \text{ for some } c \in \mathbb{R}^n \right\}$$

$$\left(\underset{m \times n}{A} \cdot \underset{n \times 1}{c} = \underset{m \times 1}{v} \right)$$

$$\operatorname{img}(A) = \underline{\operatorname{span}\{a_1, \dots, a_n\}}$$

where $A = (a_1, \dots, a_n)$.

$$A\vec{c} = (a_1 \dots a_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= (c_1 a_1 + \dots + c_n a_n)$$

$$\in \operatorname{Span}\{a_1 \dots a_n\}.$$

The image is also known as the <u>column space</u> of A.

The kernel of A is also known as the null space of A.

Prop Let A be an $m \times n$ matrix.
Then $\ker(A)$ and $\operatorname{img}(A)$ are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Pf $\operatorname{Img}(A) = \operatorname{span}\{a_1 \dots a_n\}$
Since every span is a subspace, then $\operatorname{img}(A)$ is a subspace of $\mathbb{R}^m$.

Kernel

① $0 \in \ker(A)$, $A \cdot \vec{0} = \vec{0}$. The kernel is nonempty.

② Let $\vec{v} \in \ker(A)$, $\vec{w} \in \ker(A)$.

$$A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$$
$$= \vec{0} + \vec{0} = \vec{0}.$$

Thus $\vec{v}+\vec{w} \in \ker(A)$.

③ Let $c \in \mathbb{R}$.

$$A(c\vec{v}) = c(A\vec{v}) = c \cdot \vec{0}$$
$$= \vec{0}.$$

Thus $c\vec{v} \in \ker(A)$.

So $\ker(A)$ is a subspace of $\mathbb{R}^n$.

$\square$

Superposition (section starting on pg 110
(pg 106) is not on the exam)

2.5: 105 — 109 is on the exam.

Superposition is a fancy word for
linear combination.

Superposition principle says that
if v_1 and v_2 are sol'ns to
the system $A\vec{x} = 0$.
Then so is any linear combination
of v_1 and v_2.

(Equivalent to the fact that
$\ker(A)$ is a subspace.)

If $\vec{v}_1 \in \ker(A)$ and $\vec{v}_2 \in \ker(A)$.

then so is $c_1 v_1 + c_2 v_2$.

(Same principle as in diff eq.)

Computations

Since $\ker(A) = \left\{ \begin{array}{c} \text{all solutions of} \\ \text{the linear system} \\ A\vec{x} = 0 \end{array} \right\}$,

then if we want to compute the kernel of A, all we have to do is row reduce the system

$$(A \mid 0).$$

But $\vec{0}$ is not necessary.

Any row operation on $\vec{0}$ just gives back the $\vec{0}$ column again.

So you can really just row reduce A.

<u>Ex</u>: Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Find a basis for $\ker(A)$.

First we row reduce A.

$$\begin{pmatrix} 1 & 0 & -1 \\ \boxed{2} & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{-2r_1+r_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_3} \begin{array}{c} \begin{array}{ccc} & & z \end{array} \\ \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

2 pivots

free !

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x - z = 0$$
$$y + z = 0$$
$$0 = 0$$

z is a free variable.

$$x = z$$
$$y = -z$$
$$z = z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} z.$$

Recall $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ solves

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \, !$$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a general vector in $\ker(A)$.

Thus $\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Thus $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis of the kernel.

So solving for $\vec{x}$ using the free variables spits out the basis vectors for $\ker(A)$.

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + 0 - 1 \\ 2 - 1 - 1 \\ 0 - 1 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So finding a basis for the kernel is the same as row reducing A and solving

$$A\vec{x} = \vec{0}.$$

Prop Let A row reduce to same matrix U.

Then $\ker(A) = \ker(U)$.

Pf If $A \longrightarrow U$, we know that $A\vec{x} = \vec{b}$ and $U\vec{x} = \vec{b}$ has the same sol'ns.

Let $\vec{b} = 0$. Then

$$A\vec{x} = \vec{0} \text{ and } U\vec{x} = \vec{0}$$

have the same sol'ns.

$$\ker(A) = \ker(U).$$

Slogan: Row reducing does not affect the kernel.

But what does row reduction do to the image?

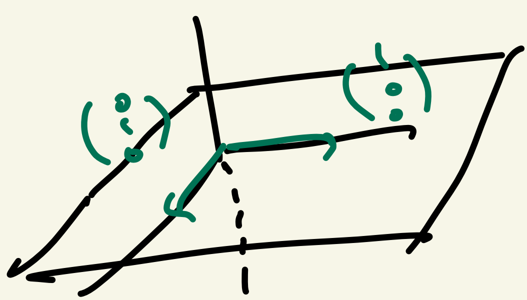
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

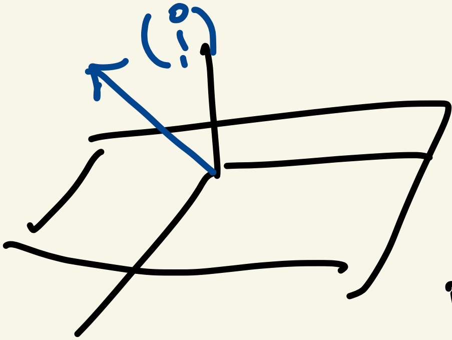
$$\mathrm{Img}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

$$A \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. = U$$

$$\mathrm{Img}(U) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \cancel{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} \right\}$$

$$= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = xy\text{-plane in } \mathbb{R}^3$$





$$\begin{pmatrix}0\\1\\1\end{pmatrix} \in \operatorname{Img}(A)$$

But $\begin{pmatrix}0\\1\\1\end{pmatrix} \notin \operatorname{img}(U)$.

$\operatorname{Img}(A) \neq \operatorname{img}(U)$ if $A \xrightarrow{\text{rr.}} U$.

But what we can say?

Notice that

$$-1u_1 + u_2 = u_3$$

$$-1\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

The same relationship is true in A.

$$-1\begin{pmatrix}1\\2\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

$$\operatorname{Img}(A) = \operatorname{span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

The linear relationships between the columns of U still hold for A.

$$\Updownarrow$$

$$c_1 u_1 + \dots + c_n u_n = 0$$

$$\left(U = (u_1 \dots u_n) \right)$$

$$U \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

In the Ex

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \ker(U) = \ker(A)$$

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{0} \Rightarrow c_1 a_1 + \dots + c_n a_n = 0$$

$$(-1)\,a_1 + (1)\,a_2 = a_3$$

$$a_1 - a_2 + a_3 = 0$$

$$(1)\,a_1 + (-1)\,a_2 + (1)\,a_3 = 0$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

We already knew $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in \ker(A)$.

So compute the $\ker(A)$ tells you the dependencies between the columns of A.

Rank - Nullity Thm

Def The rank of a matrix A is the # of leading 1's in it reduced row echelon form

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$rk(A) = 2.$

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad rk(I) = 3.$$

$$rk(A) \leq \min \left\{ \begin{array}{l} \text{\# of rows,} \\ \text{\# of columns} \end{array} \right\}$$

Rank - Nullity Thm:

Let A be $m \times n$ matrix.

Then $\boxed{rk(A) + \dim(\ker(A)) = n.}$

rank + dimension of kernel = # of columns

leading 1's + dim of kernel = # of columns.

Pf Given a matrix A, let U be the corresponding rref of A.

A, U both have n columns.

$\ker(A) = \ker(U)$ so $\dim(\ker(A)) = \dim(\ker(U))$.

Finally, U is the rref of itself, so $\mathrm{rk}(A) = \mathrm{rk}(U)$.

So it suffices to show that

$$\mathrm{rk}(U) + \dim(\ker(U)) = \overset{n}{\overset{\shortparallel}{\#}} \text{ of columns of } U$$

\# of leading 1's

\# of columns

$n - \mathrm{rk}(U) =$ \# of column $-$ \# leading 1's.

Every leading 1 is in a different column, and every column w/out a leading 1 is a <u>free</u> <u>column</u>.

$$n - \text{rk}(u) = \# \text{ of columns} - \# \text{ 1's}$$

$$= \# \text{ free columns}$$

$$= \# \text{ of free variables}$$

$$= \dim(\ker(u)).$$

So every free column

$\rightsquigarrow$ free variable

$\rightsquigarrow$ independent vector in the kernel.

$$\text{rk}(u) + \dim(\ker(u)) = n$$

$$\Longrightarrow \text{rk}(A) + \dim(\ker(u)) = n.$$

$\square$

Ex: $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Show that rk(2)

2

$U = \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\dim(\ker(A)) = 1$

since $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis.

$2 + 1 = 3 = \#.$

$2 + 1 = 3$

$\text{rank}(A) + \dim(\ker(A)) = \#$ of columns

Thm $\operatorname{rk}(A) = \dim(\operatorname{img}(A))$.

$= \#$ of linearly independent columns of of A

Pf:

$$A = (a_1 \cdots a_n)$$

$$U = \begin{pmatrix} 1 & * & * & * & 0 & * & * & * & 0 & & & 0 \\ 0 & & & & \boxed{1} & * & * & * & 0 & & & 0 \\ 0 & & & & & & & & 1 & & & 0 \\ 0 & & & & & & & & & \ddots & & 0 \\ 0 & & & & & & & & & & & 1 \end{pmatrix}$$

Let's say WLOG $v_1 \cdots v_k$ have leading 1's. The the corresponding columns in A are independent.

$\square$

$\dim(\operatorname{img}(A)) + \dim(\ker(A)) = \#$ of columns

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So a_1, a_2 are indep

and a_3 depends on a_1 and a_2.

Ex

$A \longrightarrow$ Say A now reduces to

$$\begin{array}{c} \begin{pmatrix} 1 & \boxed{-1} & 0 & -3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} \quad x & \ y & \ z & \ w \end{array} \end{array} \rightsquigarrow$$

$A = (a_1 \ a_2 \ a_3 \ a_4)$.

Which of these are dependent and independent?

$$a_1 = -a_2$$

$$a_4 = -3a_1 + 5a_3$$

a_1 and a_3 correspond to leading 1's, so they're independent.

$\text{rank}(A) = 2$

$2 + 2 = 4$

$\dim(\ker(A)) = 2$

$\#$ of columns $= 4$

Let's compute a basis for kernel of A.

$x - y + (-3)w = 0$

$z + 5w = 0$

y, w are free

$x = y + 3w$

$z = -5w$

The vectors in the kernel have the form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} y + 3w \\ y \\ -5w \\ w \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} y + \underline{\begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \end{pmatrix}} w$$

Basis for kernel $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$

Definitely know how to compute $\ker(A)$ and $\operatorname{img}(A)$ and independent and dependent columns.

<u>Thm</u> Let A be an $n \times n$ matrix. Then, the following are equivalent.

(1) A is invertible (A^{-1} exists)

(2) A has n pivots (n leading 1's) $\operatorname{rk}(A) = n$

(3) The columns of A form a basis of $\mathbb{R}^n$

(4) $\ker(A) = 0$

(5) $\operatorname{img}(A) = \mathbb{R}^n$

(6) $A\vec{x} = \vec{b}$ has a unique sol'n $\forall \vec{b} \in \mathbb{R}^n$. ($x = A^{-1}b$)

(7) $\det A \neq 0$.

Notes: If $\ker(A) = 0$, then $\text{img}(A) = \mathbb{R}^n$.

$\dim(\ker(A)) + \dim(\text{img}(A)) = n$.

If $\ker(A) = \{\vec{0}\}$, then a convention is that $\dim(\ker(A)) = 0$

$0 + \dim(\text{img}(A)) = n$.

$\dim(\text{img}(A)) = n$.

$\text{img}(A) \subseteq \mathbb{R}^n$

$\dim(\text{img}(A)) = \dim(\mathbb{R}^n)$

$\Longrightarrow \text{img}(A) = \mathbb{R}^n$.

$(*)$

What to know from today!

- rank - nullity

- leading columns in rref

$\updownarrow$

independent columns in A

- Free columns in rref

$\updownarrow$

dependent columns in A

- $rk(A)$ = # of leadings = $\dim(\operatorname{im}(A))$

= # independent columns of A

② From study guide

Find the permuted LDV decomposition of the matrix $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix}$.

Recall: $PA = LU$

permutation matrix

- all swaps during rr.

unilower Δ

- all $cr_i + r_j$ steps

upper Δ matrix you get after rr.

(rref requires back sub. don't do that when find U)

$U = DV$ D is diagonal

V is uni upper Δ

encodes all steps $r_i \to cr_i$.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\xrightarrow{\text{swap } r_1 r_3}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & \boxed{1} & 2 \end{pmatrix} \qquad P = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}\underline{r_2}+\underline{r_3}}$$

$$U = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad P = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix}$$

So $P \quad A \quad = \quad L \quad U$

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$U = DV$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & \boxed{2} & 3 \\ 0 & 0 & \boxed{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & \frac{1}{2} \end{pmatrix}^{=D} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}^{=V}$$

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$P \quad A \quad = \quad L \quad D \quad V$

(a). $A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\vec{x}^T A x$

$(1 \times \cancel{2})(\cancel{2} \times \cancel{2})(\cancel{2} \times 1) = \underline{1 \times 1}$

$$= (x \quad y) \underbrace{\begin{pmatrix} \boxed{-1} & \boxed{3} \\ \boxed{1} & \boxed{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

x^2 xy xy y

$$= (x \quad y) \begin{pmatrix} -x + 3y \\ x + 2y \end{pmatrix}$$

$$= x(-x + 3y) + y(x + 2y)$$

$$= -x^2 + (3+1)xy + 2y^2$$

⑤ V is a v.s and

U, W are subspaces.

Show that $U \cap W$ is also a subspace.

1. $U \cap W \neq \emptyset$.
2. Closed under addition
3. Closed under scalar multiplication

---

1. Since U, W are subspaces, then $\vec{0} \in U$ and $\vec{0} \in W$.

   (Look at Def 2.8)

   $\Rightarrow \vec{0} \in U \cap W$.

$V = \mathbb{R}^2$

let $W = \{(x,y) \mid x^2+y^2=1\}$

$(0,0) = \vec{0} \notin W$

since $0^2+0^2 \neq 1$

Since all subspaces contain $\vec{0}$

then W is not a subspace.

---

2 Let $\underline{\vec{v}} \in \underline{U \cap W}$ and $\vec{w} \in U \cap W$.

Consider $\vec{v} + \vec{w}$. Since U is a subsp., it is closed under addition. $v \in U$, $w \in U$

$\Rightarrow$ $\underline{\vec{v} + \vec{w} \in \underline{U}}$. Similarly $\vec{v} + \vec{w} \in \underline{W}$ as well.

then $\vec{v} + \vec{w} \in U \cap W$.

③ Let $c \in \mathbb{R}$. $\vec{v} \in U \cap W$.

In particular $\vec{v} \in U$.

$\boxed{c\vec{v} \in U}$ also because U is a subspace.

On the other hand $\vec{v} \in W$

so $\boxed{c\vec{v} \in W}$ as well.

$\Rightarrow$ $c\vec{v} \in U \cap W$.

So $U \cap W$ is closed under scalar mult.

⑥ $V = M_{n\times n}(\mathbb{R})$

the "vectors" are now matrices.

$$tr : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

$$tr\begin{pmatrix} \boxed{1} & 2 & 3 \\ -1 & \boxed{0} & 5 \\ 1 & 2 & \boxed{5} \end{pmatrix} = 1+0+5 = 6$$

Let $W = \{A \mid tr(A) = 0\}$.

① $W \neq \phi$ (Pick $\vec{0}$ element)

② closed under addition

③ closed under scal. mult.

① $0 = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

$\operatorname{tr}(0) = 0 + \dots + 0$

$0 \in W.$

② Let $A, B \in W$ i.e. $\left.\begin{array}{l} \operatorname{tr}(A) = 0 \\ \operatorname{tr}(B) = 0 \end{array}\right\}$

Want to show $\underline{\operatorname{tr}(A+B) = 0}$

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n} (A+B)_{ii}$$

$$= \sum_{i=1}^{n} (A)_{ii} + (B)_{ii}$$

$$= \sum_{i=1}^{n} (A)_{ii} + \sum_{i=1}^{n} (B)_{ii}$$

$$= 0 + 0 = 0$$

③ Let $c \in \mathbb{R}$, $A \in W$ $(tr(A) = 0)$

Want to show $tr(cA) = 0$.

$$tr(cA) = \sum_{i=1}^{n} (cA)_{ii}$$

$$= \sum_{i=1}^{n} c(A)_{ii} = c\left(\sum_{i=1}^{n} (A)_{ii}\right)$$

$$= c \cdot 0 = 0 \qquad \square$$

$u'' + u' + u = 0$

Solutions form a subspace

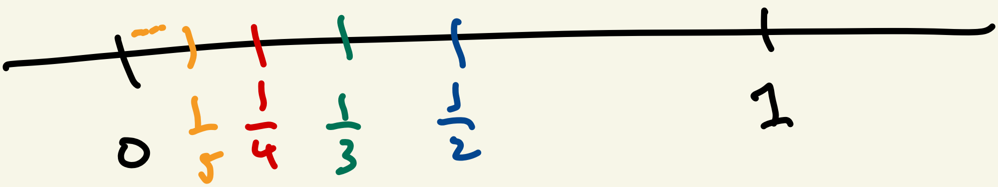
$u_1 \text{---} \longrightarrow \underline{u}$

Convergent Sequences

$0, 1, 2, 3, 4, \ldots$ does not converge

$\longrightarrow$ this pattern doesn't approach anything

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ does converge



$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \longrightarrow 0$

$a_n \longrightarrow L = \lim a_n$ $\qquad \{\varepsilon = \frac{1}{10^9}\}$

if $\forall \varepsilon > 0 \; \exists N$ s.t. $\forall n > N$ $\qquad N = 10^9 + 1$

$|a_n - L| < \varepsilon$.

2.2.27 functions

Reduced row echelon form

columns in rref either have a leading 1 or not.

$$
\begin{array}{ccccc}
x & y & z & w & u \\
\end{array}
$$
$$
\left(\begin{array}{ccccc}
1 & \boxed{-3} & 0 & \boxed{4} & 0 \\
0 & \boxed{0} & 1 & \boxed{-1} & 0 \\
0 & \boxed{0} & 0 & \boxed{0} & 1
\end{array}\right)
$$

Every column w/out a leading 1 is a free column.

Can't solve for y, w. But at best solve for x, z, u in terms y, w.

$$\rightarrow \begin{pmatrix} 1 & 0 & \boxed{-1} & 0 \\ 0 & 1 & \boxed{3} & 0 \\ 0 & 0 & \boxed{0} & 1 \\ 0 & 0 & \boxed{0} & 0 \end{pmatrix}$$

Then we would know

$$-1a_1 + 3a_2 = a_3$$

But a_4 would be independent

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad a_4 = -a_1 + 2a_2 + 5a_3$$

5, 3x and 1+x.

$$\frac{1}{5}(5) + \frac{1}{3}(3x) = 1+x. \quad ✓$$

$e^{5x} + \sin(x),\ \cos^2(x),\ \tan(x)$??

Wronskian ✗

Elementary permutation matrix

$\updownarrow$

$$\text{swap}(r_i r_j) \qquad \begin{array}{cc} & \begin{matrix} & & ij & \end{matrix} \\ ji & \begin{bmatrix} 1 & & 1 & \\ & & \ddots & \\ & 1 & & \ddots \end{bmatrix} \end{array}$$