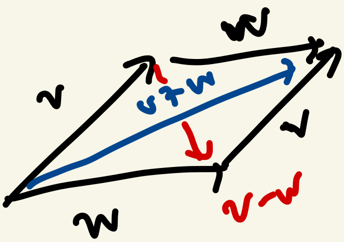
Exam 1 Recap

- Median 54 (St Dev 16)

Chapter 3



Geometric Applications

§ 3.1 Inner Products

$\mathbb{R}^n$ $(v_1, \ldots, v_n)$ $\xrightarrow{\text{generalize}}$ vector spaces

dot product on $\mathbb{R}^n$ $\rightsquigarrow$ inner product

<u>Def</u> Let V be a real vector space. Then an <u>inner product</u> on V is a pairing $\langle -,- \rangle$ which outputs a real number such that

(i) • $\langle c\vec{u} + d\vec{v}, \vec{w} \rangle = c\langle \vec{u}, \vec{w} \rangle + d\langle \vec{v}, \vec{w} \rangle$

• $\langle \vec{u}, c\vec{v} + d\vec{w} \rangle = c\langle \vec{u}, \vec{v} \rangle + d\langle \vec{u}, \vec{w} \rangle$

(bilinearity)

(ii) $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$ (symmetry)

(iii) If $\vec{v} \neq \vec{0}$, $\langle \vec{v}, \vec{v} \rangle > 0$ and $\langle \vec{0}, \vec{0} \rangle = 0$. (positivity) (positive-definiteness)

A pairing of vectors, V real vector space

$$\underbrace{\langle -, - \rangle}_{\text{function which takes 2 vectors as inputs and outputs a real number.}} : V \times V \longrightarrow \mathbb{R}$$

In order for a pairing to be an inner product, it must satisfy the three rules: bilinearity, symmetry, positivity.

Ex $V = \mathbb{R}^n$

Define $\langle \vec{v}, \vec{w} \rangle = v_1 w_1 + \dots + v_n w_n$

$$= \sum_{i=1}^{n} v_i w_i = \vec{v} \cdot \vec{w}.$$

(aka the dot product)

The dot product is an inner product.

(i) $(c\vec{u} + d\vec{v}) \cdot \vec{w}$

$$= \sum_{i=1}^{n} (cu_i + dv_i) w_i$$

$$= \sum_{i=1}^{n} c(u_i w_i) + d(v_i w_i)$$

$$= c \sum_{i=1}^{n} u_i w_i + d \sum_{i=1}^{n} v_i w_i$$

$$= c(u \cdot w) + d(v \cdot w)$$

Second component is the same proof.

$$\vec{u} \cdot (c\vec{v} + d\vec{w})$$
$$= c(u \cdot v) + d(u \cdot w)$$

(ii)
$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$
$$= \sum_{i=1}^{n} w_i v_i$$
$$= \vec{w} \cdot \vec{v}.$$

(iii) If $\vec{v} \neq \vec{0}$ then
$$\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 > 0$$ since one of the $v_i \neq 0$ and squares are always positive.

$$\vec{0}\cdot\vec{0} = \sum_{i=1}^{n} 0\cdot 0 = 0.$$

Therefore the dot product is an example of an inner product.

Ex $V = \mathbb{R}^2$ $(n=2)$

Two examples

$$\langle v, w\rangle = 3v_1w_1 + \underbrace{5v_2w_2}_{\text{heigher weight}}$$

This is an inner product.

$$\langle cu+dv, w\rangle$$

$$= 3(cu_1+dv_1)w_1 + 5(cu_2+dv_2)w_2$$

$$= c\overbrace{(3u_1w_1 + 5u_2w_2)}$$

$$+ d\underbrace{(3v_1w_1 + 5v_2w_2)} =$$

$$c\langle u,w\rangle + d\langle v,w\rangle$$

- $\langle v, w \rangle = 3v_1w_1 + 5v_2w_2$

$$= 3w_1v_1 + 5w_2v_2$$

$$= \langle w, v \rangle$$

- $\langle v, v \rangle = 3v_1^2 + 5v_2^2 > 0$ if $v \neq 0$

These can be any positive number.

$\langle 0, 0 \rangle = 3 \cdot 0 + 5 \cdot 0 = 0$

So $\langle v, w \rangle = \underline{3}v_1w_1 + \underline{5}v_2w_2$ is an inner product.

Another Example on $\mathbb{R}^2$.

$\langle v, w \rangle = v_1w_1 - \underline{v_1w_2} - \underline{v_2w_1} + 4v_2w_2$

- Bilinearity and Symmetry

✓

$\langle v, v \rangle$

$= v_1^2 - v_1 v_2 - v_2 v_1 + 4v_2^2$

$= \underline{v_1^2 - 2v_1 v_2 + v_2^2} + 3v_2^2$

$= (v_1 - v_2)^2 + 3v_2^2 \geq 0$

If $v \neq 0$ then $\langle v, v \rangle > 0$

If $\langle 0, 0 \rangle = 0$. ✓

Def A vector space w/ an inner product is called an <u>inner</u> <u>product</u> <u>space</u>.

Def Given $\vec{v} \in V$ (V is an inner product space)

The <u>norm</u> of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{\langle v, v \rangle} \geq 0$$

If $\langle v, w \rangle = v \cdot w$

then $\|v\| = \sqrt{v \cdot v}$

$$= \sqrt{\sum v_i^2}$$

$$= \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Usual idea of magnitude.

Prop Let V be an inner product space. Then $\langle -, - \rangle$ satisfies

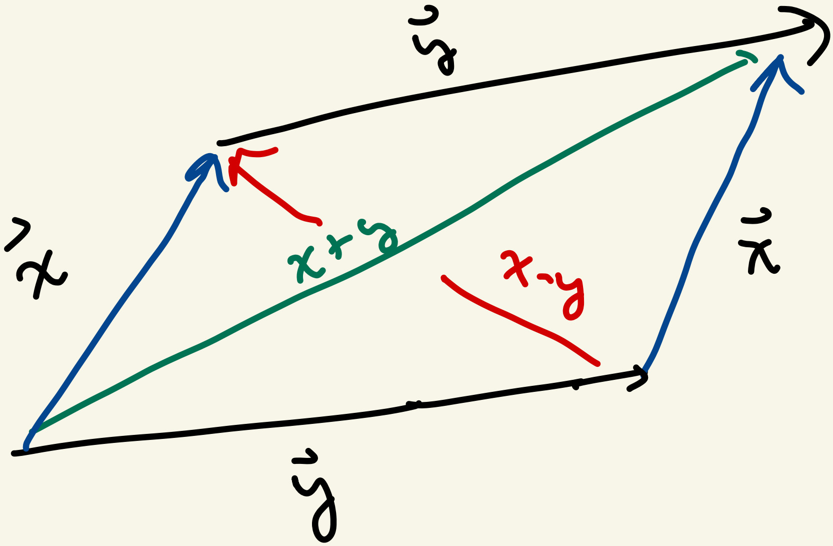
- $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$

  (parallelogram identity)

- $4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2$

  (polarization identity)

We'll see this later.



$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

$\|x\|$ = length of $\vec{x}$

$\|y\|$ = length of $\vec{y}$

$\|x+y\|$ = length of diagonal

$\|x-y\|^2$ = length of anti-diagonal

This is why studying inner product spaces is a version of geometry.

More Examples

Let $V = C^0[a,b]$, where this is the vector space of continuous functions on $[a,b]$.

$(f : [a,b] \longrightarrow \mathbb{R})$

Define $\langle f, g \rangle$

$$= \int_a^b f(x) g(x) \, dx .$$

This is an inner product.

- $\langle cf + dg, h \rangle$

$$= \int_a^b \big(cf(x) + dg(x)\big) h x \, dx$$

$$= c\int_a^b f(x) h(x) dx + d\int_a^b g(x) h(x) dx$$

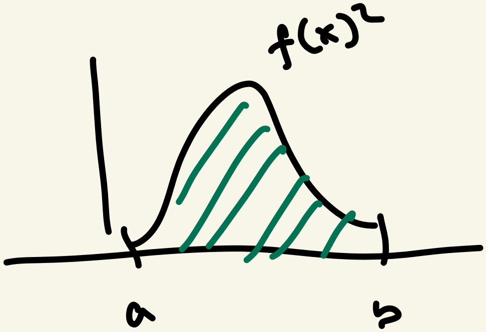
$$= c\langle f, h \rangle + d\langle g, h \rangle .$$

- $\langle f, g \rangle = \int_a^b f(x)\,g(x)\,dx$

  $= \int_a^b g(x)\,f(x)\,dx = \langle g, f \rangle$

- If $f \neq 0$

  $\langle f, f \rangle = \int_a^b f(x)^2\,dx > 0$



$\langle 0, 0 \rangle = \int 0\,dx$

$= 0$

$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2\,dx}$

L^2- norm

- Example Calculation $V = C^0[0,\pi]$

$\langle \sin(x), \cos(x) \rangle$

$$= \int_0^{\pi} \sin(x)\cos(x)\,dx$$

$\frac{d}{dx}\sin(x)$

$= \cos(x)$

$$= \int u\,du = \left(\frac{1}{2}u^2\right)$$

$$= \left(\frac{1}{2}\sin(x)^2\right)_0^{\pi}$$

$$= \frac{1}{2}\left(\sin(\pi)^2 - \sin(0)^2\right)$$

$$= \frac{1}{2}(0^2 - 0^2) = \underline{0}$$

$(v \cdot w = 0 \Rightarrow v \perp w)$

In $C^0[0,\pi]$, $\sin(x)$ and $\cos(x)$ are $\perp$.