For the record,

Exam 2 is on 7/10 !

Loose Ends from §3.1.

Pf of Polarization id is on the HW. (3.1.12)

Prop Let V be an inner product space. Then

$$2\|v\|^2 + 2\|w\|^2 = \|v+w\|^2 + \|v-w\|^2$$

Pf

$$\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$$

$$= \langle v+w, v\rangle + \langle v+w, w\rangle + \langle v-w, v\rangle - \langle v-w, w\rangle$$

$$= \langle v, v\rangle + \langle w, v\rangle + \langle v, w\rangle + \langle w, w\rangle + \langle v, v\rangle - \langle w, v\rangle - \langle v, w\rangle + \langle w, w\rangle$$

$$= 2\langle v, v\rangle + \cancel{\langle v, w\rangle} + \cancel{\langle v, w\rangle} + 2\langle w, w\rangle - \cancel{\langle v, w\rangle} - \cancel{\langle v, w\rangle}$$

$$= 2\langle v, v\rangle + 2\langle w, w\rangle$$

$$= 2\|v\|^2 + 2\|w\|^2 \qquad \square$$

This works for any inner product, not just dot product!

Recall the L^2 inner product.

$V = C^0[a,b]$.

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

$$\|f\| = \sqrt{\int_a^b f(x)^2 \, dx}$$

This is an inner product.

- $\langle f, f \rangle > 0$ for $f \neq 0$ and

  $\langle 0, 0 \rangle = 0$. (positivity axiom)

$$\begin{cases} \text{\underline{Prop} let } g(x) \neq 0 \text{ and } g(x) \geq 0. \\ \text{on } [a,b]. \quad \text{If } g(x) \text{ is cts} \\ \text{then } \int_a^b g(x)\,dx > 0. \end{cases}$$

From Calc 2: If $g(x) \geq 0$

$$\Rightarrow \int_a^b g(x)\,dx \geq 0.$$

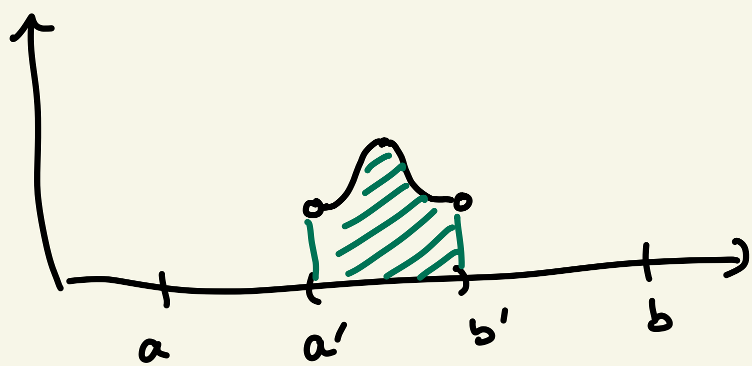
$> ?$

If $g(x) \geq 0$, $\neq 0$ and $g(x)$ cts

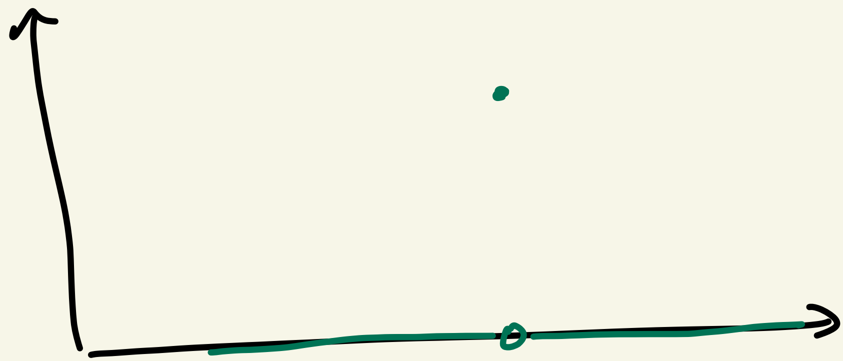
$$\text{then } \int_a^b g(x)\,dx > 0 \;!$$

(Use this for when $g(x) = f(x)^2$

or $g(x) = f(x)^2 e^{-x}$)



If $g(x)$ is not cts, then this happens.



$g(x) \not\equiv 0$ and $g(x) \geq 0$

but $\int_a^b g(x)\,dx = 0.$

Continuity is necessary!

(HW 3.1.23 abd) need this result to prove positivity

§ 3.2 Inequalities

Thm Let V be an inner product space. Let $v, w \in V$.

Then
$$|\langle v, w\rangle| \leq \|v\| \cdot \|w\| .$$

Cauchy-Schwartz Inequality

Pf If $w = 0$, then $|\langle v, w\rangle| = 0$ and $\|v\| \, \|w\| = 0$ so the inequality holds.

If $w \neq 0$, then we can consider the following.

Let $t \in \mathbb{R}$. Calculate the norm $\|v + tw\|^2$.

$$0 \le \|v + tw\|^2 = \langle v + tw, v + tw \rangle$$

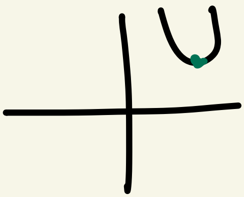
$$= \langle v, v \rangle + \langle tw, v \rangle + \langle v, tw \rangle + \langle tw, tw \rangle$$

$$= \langle v, v \rangle + t \langle v, w \rangle + t \langle v, w \rangle + t^2 \langle w, w \rangle$$

$$= \|v\|^2 + 2t \langle v, w \rangle + t^2 \|w\|^2$$

This is a quadratic polynomial in the variable t.

Since $\|w\|^2 > 0$ $(w \neq 0)$



It's minimum value ≥ 0.

We'll get the closest inequality from $\boxed{0 \le \|v + tw\|^2}$ when t makes this parabola at it's minimum value.

Therefore we minimize the polynomial

$$p(t) = \|w\|^2 t^2 + 2\langle v, w\rangle t + \|v\|^2$$

when $p(t)$ is has a min, $p'(t) = 0$.

$$p'(t) = 2\|w\|^2 t + 2\langle v, w\rangle = 0$$

Solving, $\boxed{t = \dfrac{-\langle v, w\rangle}{\|w\|^2}}$.

Plug in $t = \dfrac{-\langle v, w\rangle}{\|w\|^2}$ back into parabola.

$$0 \le \left\| v + \left(-\frac{\langle v,w\rangle}{\|w\|^2} \right) w \right\|^2$$

$$= \|v\|^2 + 2\langle v,w\rangle \left(\frac{-\langle v,w\rangle}{\|w\|^2} \right) + \|w\|^2 \left(\frac{-\langle v,w\rangle}{\|w\|^2} \right)^2$$

$$= \|v\|^2 - \frac{2\langle v,w\rangle^2}{\|w\|^2} + \frac{\langle v,w\rangle^2}{\|w\|^2}$$

$$= \|v\|^2 - \frac{\langle v,w\rangle^2}{\|w\|^2} \geq 0$$

$$\|v\|^2 \geq \frac{\langle v,w\rangle^2}{\|w\|^2}$$

$$\sqrt{\langle v,w\rangle^2} = |\langle v,w\rangle|$$

$$\|v\| \cdot \|w\| \geq |\langle v,w\rangle|.$$

$\square$

Ex let $V = \mathbb{R}^3$ w/ dot product

let $v = (v_1 v_2 v_3)^T$

$w = (w_1 w_2 v_3)^T$.

$$|\langle v, w \rangle| \le \|v\| \, \|w\|$$

$$\overbrace{|v_1 w_1 + v_2 w_2 + v_3 w_3|}^{\text{fewer positive terms}}$$

$$\le \underbrace{(v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2)}_{\text{lots of positive terms}}$$

Ex $V = C^0[a,b]$

$$|\langle f, g \rangle| \le \|f\| \, \|g\|$$

$$\left| \int_a^b f(x) g(x) \, dx \right| \le \sqrt{\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx}$$

(OG Cauchy-Schwartz)

First use of C-S is to $\cos\theta$.

$$|\langle v,w\rangle| \leq \|v\| \, \|w\|$$

$$\implies \frac{|\langle v,w\rangle|}{\|v\| \, \|w\|} \leq 1$$

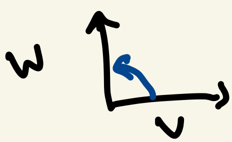
$\cos\theta$

$$\implies -1 \leq \boxed{\frac{\langle v,w\rangle}{\|v\| \, \|w\|}} \leq 1$$

$\cos\theta$, $\sin\theta$ are also bounded by $-1, 1$.

$V = \mathbb{R}^2$ w/ dot product

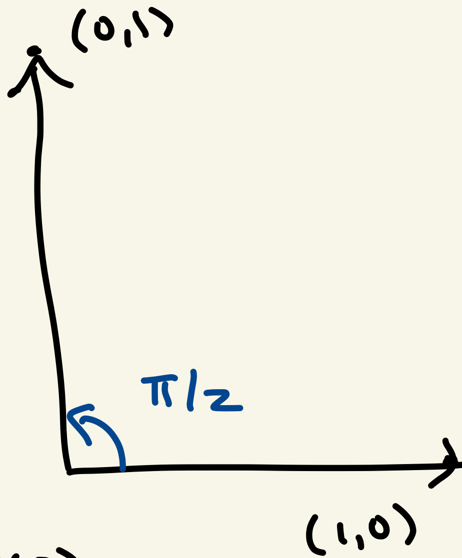
$v = (1,0)$ and $w = (0,1)$



$$\langle v,w\rangle = (1)(0) + (0)(1)$$

$$= 0$$

$$\Longrightarrow \frac{\langle v,w\rangle}{\|v\|\,\|w\|} = \frac{0}{1\cdot 1} = \boxed{0}$$

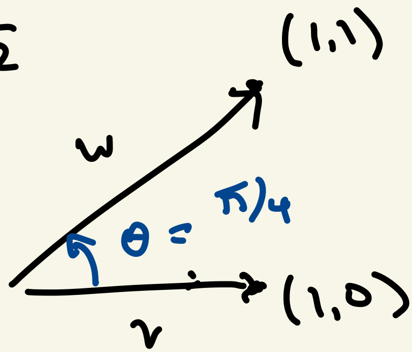


θ is the angle between v and w

$$\cos\theta = \cos(\pi/2) = \boxed{0}$$

$$\frac{\langle v,w\rangle}{\|v\|\,\|w\|} = \frac{(1)(1)+(1)(0)}{1\cdot\sqrt{2}}$$

$$= \boxed{\frac{1}{\sqrt{2}}}$$



$$\cos(\pi/4) = \boxed{\frac{1}{\sqrt{2}}}$$

Def Let V be an inner product space.

Define the angle θ between $v, w \in V$ by the formula

$$\theta = \cos^{-1}\left(\frac{\langle v, w\rangle}{\|v\| \, \|w\|}\right)$$

Note: $\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$

is $\cos^{-1}(0) = \pi/2, \; 3\pi/2, \; -\pi/2$???

Restrict $\cos^{-1}$ to values between $0, \pi$.

$\cos^{-1}(0) = \frac{\pi}{2}$ $\quad \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

let $V = C^0[0,1]$. L^2 inner product

compute "angle" between $f(x) = x$, $g(x) = x^2$.

$$\|f\| = \sqrt{\int_0^1 f(x)^2\,dx}$$

$$\theta = \cos^{-1}\left(\frac{\langle f, g\rangle}{\|f\|\,\|g\|}\right)$$

$$= \cos^{-1}\left(\frac{\int_0^1 x\cdot x^2\,dx}{\sqrt{\int_0^1 x^2\,dx}\sqrt{\int_0^1 x^4\,dx}}\right)$$

is a ratio of integrals

$$= \cos^{-1}\left(\frac{\int_0^1 x^3\,dx}{\sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}}}\right)$$

$$= \cos^{-1}\left(\frac{\frac{1}{4}}{\sqrt{\frac{1}{15}}}\right) = 0.25268\ldots \text{ rad.}$$

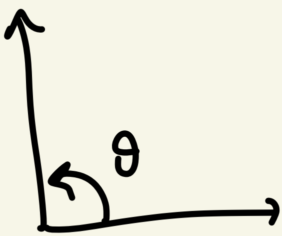
No geometric meaning. :(

Remember that in $\mathbb{R}^2$

$$\cos(\pi/2) = 0$$

Two vectors in $\mathbb{R}^2$ w/ dot product

are $v \perp w$ iff $\langle v, w \rangle = 0$.



$$\cos\theta = \frac{\langle v, w \rangle}{\|v\| \, \|w\|}$$

$$\boxed{v \perp w} \iff \theta = \pi/2$$

$$\iff \cos\theta = 0 \qquad (\theta \in [0, \pi])$$

$$\iff \frac{\langle v, w \rangle}{\|v\| \, \|w\|} = 0$$

$$\iff \boxed{\langle v, w \rangle = 0}$$

Take this result and use it for general inner products.

<u>Def</u> Let V be an inner product space. Let $v, w \in V$.

Then v is <u>orthogonal</u> to w (written $v \perp w$) if $\langle v, w \rangle = 0$.

(orthogonal is fancy for perpendicular)

Note: <u>orthogonal</u> <u>depends</u> on <u>inner product</u>!

<u>Ex</u> $(1,0,1)^T$ $(5,-3,-5)^T$ $\in \mathbb{R}^3$ w/ dot prod.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix} \right\rangle = (1)(5) + (0)(-3) + (1)(-5) = 0$$

But if we consider $\mathbb{R}^3$ w/

$\langle v, w \rangle = v_1 w_1 + 2 v_2 w_2 + 3 v_3 w_3$.

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix} \right\rangle =$$

$$(1)(5) + 2(0)(-3) + 3(1)(-5)$$

$$= 5 - 15 = -10$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$ no longer orthogonal

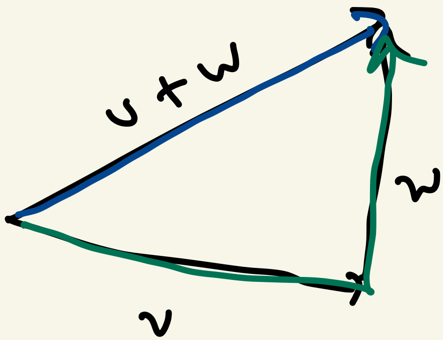
But $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -3 \\ -5/3 \end{pmatrix}$ are

orthogonal!

Triangle Inequality

Thm Let V be an inner product space. Let $v, w \in V$.

Then $\|v+w\| \le \|v\| + \|w\|$



(Diagonals are shorter!)

Pf
$$\|v+w\|^2 = \langle v+w, v+w\rangle$$
$$= \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$
$$= \|v\|^2 + 2\langle v,w\rangle + \|w\|^2$$
C-S $$\le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$$
$$= (\|v\| + \|w\|)^2$$

Taking sq root of both sides

$$\|v+w\|^2 \le (\|v\| + \|w\|)^2$$

$$\|v+w\| \le \|v\| + \|w\|. \quad \square$$

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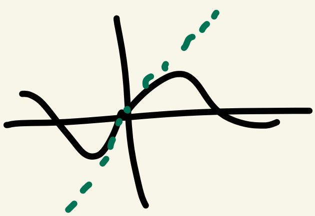
1. Let $V = C^0[-1,1]$. Let f be odd and g even.

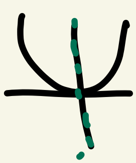
Recall f is odd function when



$$f(-x) = -f(x).$$

A function is even when

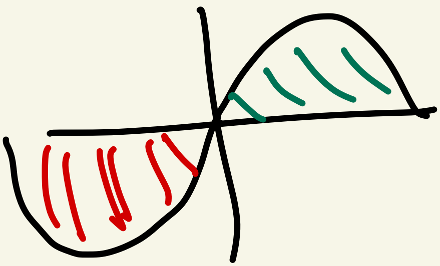
$$g(x) = g(-x).$$



Claim: $f(x)g(x)$ is odd.

Claim: If $h(x)$ is odd then

$$\int_{-a}^{a} h(x)\,dx = 0. \checkmark$$



$$f(-x)g(-x) = \big(-f(x)\big)\big(g(x)\big)$$

$$= -\big(f(x)g(x)\big) \text{ so}$$

$f \cdot g$ is odd.

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx = 0$$

since $f(x)g(x)$ is odd.

2. Let A be a matrix and $v \in \ker(A)$.

Show that $v \perp a_{i*}$ where a_{i*} is a row of A.

By definition

$$Av = 0.$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \cdot v \\ a_2 \cdot v \\ \vdots \\ a_n \cdot v \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ $a_1 \cdot v = 0$
$a_2 \cdot v = 0$ etc.

$v \cdot a_i = 0 \implies v \perp a_i \ \forall i.$

3. Orthogonal matrices.

A matrix is called orthogonal if $A^{-1} = A^T$. (A is $n \times n$)

Let $A = (v_1 \dots v_n)$ (v_i is the i^{th} column)

$$\begin{cases} v_i \cdot v_j = 0 & \text{if } i \neq j \\ v_i \cdot v_i = 1 \end{cases}$$

Ex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(Recall: columns are orthogonal to each other, and they're unit vectors.)

Show that A is orthogonal.
(i.e. that $A^{-1} = A^T$).

Recall that in general

$$(AB)_{ij} = a_{i*} \cdot b_{*j}$$

ith row   jth column

$$\boxed{A^T A} = I$$ (therefore $A^T = A^{-1}$)

$(A^TA)_{ij} =$ ith row of A^T · jth column of A

$$= \underline{v_i \cdot v_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$A^T A = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix} = I$$

So $A^T = A^{-1}$, as desired.

$$\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \langle v_1, v_1 \rangle & & \overset{1}{} & \overset{0}{\underline{\langle v_j, v_i \rangle}} \\ & \langle v_2, v_2 \rangle & & \\ & & \ddots & \\ \underset{0}{\underline{\langle v_i, v_j \rangle}} & & & \langle v_n, v_n \rangle \end{pmatrix}$$

$$= I$$

§ 3.3 Norms

Given an inner product $\langle v, w \rangle$,

it gives rise to a norm

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

- $2\|v\|^2 + 2\|w\|^2 = \|v+w\|^2 + \|v-w\|^2$
- $\|u+w\| \leq \|v\| + \|w\|$
-

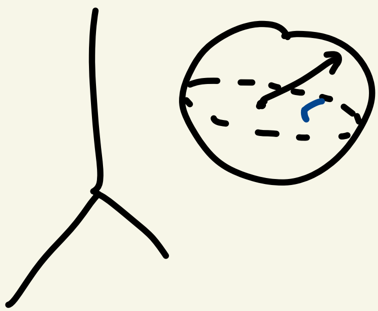
These only involve the norm.

Maybe a norm should be a concept in an of itself.

Furthermore $d(v,w) = \|v-w\|$ is a notion of distance.

(If you've taken 3283)

$$B(x, r) = \{ v \in \mathbb{R}^n \mid \|v - x\| < r \}$$



Since $\|v - w\| = d(v, w)$ this seems useful....

Norms seem independent conceptually from inner products.

Def A norm on a vector space V is any function

$\|-\| : V \longrightarrow \mathbb{R}$ s.t.

- $\|v\| \geq 0$ (positivity)
- $\|cv\| = |c| \, \|v\|$ (homogeneity)
- $\|v+w\| \leq \|v\| + \|w\|$. ($\Delta$ inequality)

Any function $\|-\| : V \longrightarrow \mathbb{R}$ w/ these properties is a norm.

Ex: let $V = \mathbb{R}^n$.

$\|v\|_1 = \sum |v_i|$ is a norm.

called the <u>L^1 norm</u> on $\mathbb{R}^n$.

- $\|v\|_1$

$$= |v_1| + |v_2| + \dots + |v_n|$$

$$\geq 0 \quad (\text{positivity} ✓)$$

- $\|cv\|_1 = |cv_1| + \dots + |cv_n|$

$$= |c| \left(|v_1| + \dots + |v_n| \right)$$

$$= |c| \, \|v\|_1 \quad ✓$$

- $\|v+w\|_1 \overset{?}{\leq} \|v\|_1 + \|w\|_1$

$$\sum |v_i + w_i| \overset{?}{\leq} \sum |v_i| + \sum |w_i|$$

Proven using any number of methods

$\|v+w\|_n$

$= \|(v_1+w_1, \dots, v_n+w_n)\|_1$

$= \sum |v_i+w_i|$ $\leq$

$\|v\|_1 + \|w\|_1 = \sum |v_i| + \sum |w_i|$

$|x+y| \leq |x| + |y|$

(This is just the Δ inequality for $\mathbb{R}^1$ as an inner product space.)

So Δ ineq. of $\|\cdot\|_1$ proven. ✓

Ex $V = \mathbb{R}^n$

$$\|v\|_\infty = \max\{|v_1|, \dots, |v_n|\}$$

This is a norm. (L^∞-norm)

- $\|v\|_\infty = \max\{|v_1|, \dots, |v_n|\} \geq 0$ ✓

- $\|cv\|_\infty = \max\{|cv_1|, \dots, |cv_n|\}$

$$= |c| \max\{|v_1|, \dots, |v_n|\}$$

$$= |c| \, \|v\|_\infty$$

- $\|v+w\| \leq \|v\| + \|w\|$

$$\Big[\max\{|v_1+w_1|, \dots, |v_n+w_n|\}\Big]$$

$$\leq \max\{|v_1|, \dots, |v_n|\}.$$

$$+ \max\{|w_1|, \dots, |w_n|\}.$$

Turns out to be true. Pf tomorrow.

<u>Ex</u> L^2- norm

$$\|v\|_2 = \sqrt{\sum |v_i|^2} = \sqrt{\sum v_i^2}$$

$$\|v\|_2 = \sqrt{v \cdot v}$$

- $\|v\|_2 \geq 0$
- $\|cv\| = |c|\,\|v\|$ ✓
- $\|v+w\| \leq \|v\| + \|w\|$ proven earlier

Claim: There's no possible inner product such that

$$\begin{cases} \|v\|_1 = \sqrt{\langle v, v\rangle} & \text{or} \\ \|v\|_\infty = \sqrt{\langle v, v\rangle} \end{cases}$$

<u>Pf</u> <u>tomorrow</u>