Loose ends from Yesterday...

$\underline{\text{Pf}}$ of the L^∞-norm on $\mathbb{R}^n$.

$$\|v\|_\infty = \max\{|v_1|, \dots, |v_n|\}$$

Pf of Δ-inequality for $\|-\|_\infty$.

$v, w \in \mathbb{R}^n$

$$\|v+w\|_\infty = \max\{|v_1+w_1|^{\Delta}, |v_2+w_2|^{\Delta}, \dots, |v_n+w_n|^{\Delta}\}$$

$$\leq \max\{|v_1|+|w_1|, \dots, |v_n|+|w_n|\}$$

WLOG that $\underline{|v_1|} + \underline{|w_1|}$ achieves the maximum.

But let $|v_i| = \max\{|v_1|, \dots, |v_n|\}$

$$|w_j| = \max\{|w_1|, \dots, |w_n|\}$$

$$|v_1|+|w_1| \leq |v_i| + |w_j| \rightsquigarrow$$

$$|v_1| + |w_1| \leq \max\{|v_1|, \dots, |v_n|\} + \max\{|w_1|, \dots, |w_n|\}$$

So in conclusion

$$\underline{\|v+w\|_\infty} \leq \max\{|v_1| + |w_1|, \dots, |v_n| + |w_n|\}$$

$$\leq \max\{|v_1|, \dots, |v_n|\} + \max\{|w_1|, \dots, |w_n|\}$$

$$= \underline{\|v\|_\infty + \|w\|_\infty}$$

Therefore the L^∞ norm satisfies the Δ-ineq.

Other loose end:

Claim: $||v||_1$ and $||v||_\infty$ do not arise from inner products.

There is no inner product $\langle -, - \rangle$ such that

$$||v||_1 = \sqrt{\langle v, v \rangle} \quad \text{or} \quad ||v||_\infty = \sqrt{\langle v, v \rangle}.$$

Every inner product gives you a norm. But not every norm comes from an inner product.

Proposition: let V be a normed v.s., i.e. V is equipped with a norm $\|-\|: V \longrightarrow \mathbb{R}$.

Then $\exists$ an inner product $\langle -,- \rangle$ s.t. $\|v\| = \sqrt{\langle v,v \rangle}$ $\forall v$ iff the norm satisfies the parallelogram identity

$$2\|v\|^2 + 2\|w\|^2 = \|v+w\|^2 + \|v-w\|^2.$$

Pf If $\|-\|$ cam from an inner product, then polarization id tells you

$$\langle v,w \rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$$

has to be the inner product.

It comes down to showing that

$$\frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$$ satisfies

the inner product axioms.

- Bilinearity only true when parallelogram identity holds
- Symmetry ✓
- Positivity ✓

$\square$.

All we have to do is show that

$\|-\|_1$ and $\|-\|_\infty$ don't

satisfy the parallelogram identity.

let $V = \mathbb{R}^2$

$$\|(v_1, v_2)\|_1 = |v_1| + |v_2|$$

$$\|(v_1, v_2)\|_\infty = \max\{|v_1|, |v_2|\}$$

$v = (1,3) \qquad v + w = (-1,7)$

$w = (-2,4) \qquad v - w = (3,-1)$

$$2\|v\|_1^2 + 2\|w\|_1^2 = 2(4)^2 + 2(6)^2$$

$$= 2 \cdot 16 + 2 \cdot 36 = 32 + 72 = 104$$

$$\|v+w\|^2 + \|v-w\|^2 = (8)^2 + (4)^2 = 64 + 16$$
$$= 80 \neq 104$$

$$2\|v\|_\infty^2 + 2\|w\|_\infty^2 = 50$$

$$\|v+w\|_\infty^2 + \|v-w\|_\infty^2 = 58 \neq 50$$

So no inner product!

§ 3.3 continued

$\|v\|_1$, $\|v\|_\infty$, $\|v\|_2$ are all examples of a more general formula.

<u>L^p-norm</u> $\|\cdot\|_p$ on $\mathbb{R}^n$

$$\|v\|_p = \left(\sum |v_i|^p \right)^{1/p}$$

$$\left(\lim_{p \to \infty} \|v\|_p = \|v\|_\infty \right)$$

This is a norm for $1 \leq p \leq \infty$.

$$\|v\|_{500} = \sqrt[500]{\sum |v_i|^{500}}$$

$$v = (1,2,3)$$

$$\|v\|_{500} = \sqrt[500]{\underbrace{1^{500} + 2^{500}}_{\text{supersmall}} + 3^{500}}$$

$$\approx \sqrt[500]{3^{500}} = 3$$

Let $V = C^0[a,b]$. There's an L^p-norm on this vector space as well.

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

$$\|f\|_\infty = \max \{ f(x) \mid a \leq x \leq b \}.$$

Unit vector and Unit Spheres.

Let V be a <u>normed</u> vector space.

Then for all $v \neq 0$, let

$$u = \frac{v}{\|v\|} = \frac{1}{\|v\|} v.$$

Then u is called the unit vector associated to v.

<u>Prop</u>: $\|u\| = 1$.

<u>Pf</u>: $$\|u\| = \left\| \frac{v}{\|v\|} \right\|$$

$$= \left| \frac{1}{\|v\|} \right| \|v\| = \frac{1}{\|v\|} \|v\|$$

$$= 1.$$

What the unit vector is depends on the norm.

$\mathbb{R}^2$ w/ L^2-norm

$v = (1,1)$

$$u = \frac{1}{\|v\|_2} v = \frac{1}{\sqrt{1+1}} v$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

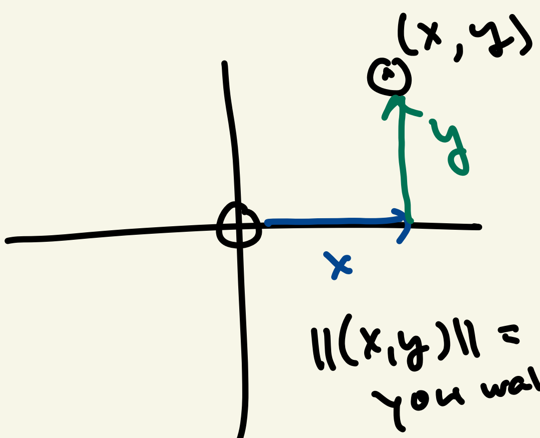
$\mathbb{R}^2$ w/ L^1-norm.

$v = (1,1)$

$$u = \frac{1}{\|v\|_1} v$$

$$= \frac{1}{1+1}(1,1)$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$



$\|(x,y)\|$ = how far you walk on "city streets".

Unit spheres for $\|\cdot\|$.

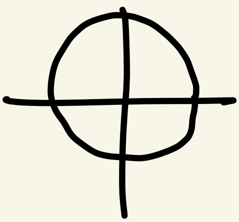
Let V be a normed vector space.

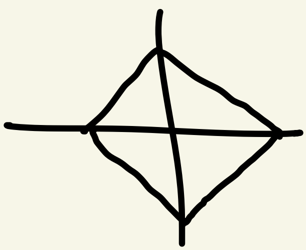
Let $S_1 = \{ v \in V \mid \|v\| = 1 \}$

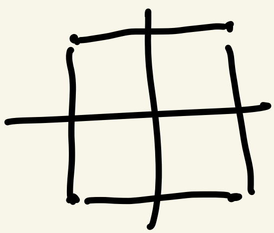
$= $ set of unit vectors.

Depending on $\|\cdot\|$, S_1 will have a different shape.

Let's fix $V = \mathbb{R}^2$.







$\|\cdot\|_2$ $\qquad$ $\|\cdot\|_1$ $\qquad$ $\|\cdot\|_\infty$

$x^2 + y^2 = 1$ $\qquad$ $|x| + |y| = 1$ $\qquad$ $\max\{|x|, |y|\} = 1$

What is the relationship between these spheres?

<u>Theorem</u>: Let $V = \mathbb{R}^n$. Then for two norms $\|-\|_a$, $\|-\|_b$, there exist constants c, d such that

$$c\|v\|_a \le \|v\|_b \le d\|v\|_a$$

$\forall v \in V$ simultaneously.

(So c, d are independent of v).

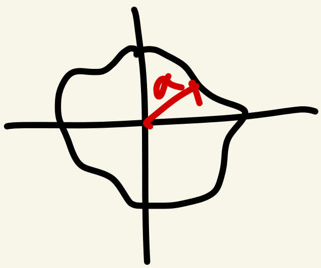
Slogan: Any two norms on $\mathbb{R}^n$ are "equivalent".

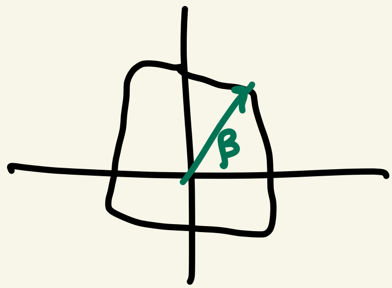
↳ stick one sphere in another

Let's fix a vector $v \in \mathbb{R}^n$.

$$\|v\|_a = \alpha$$

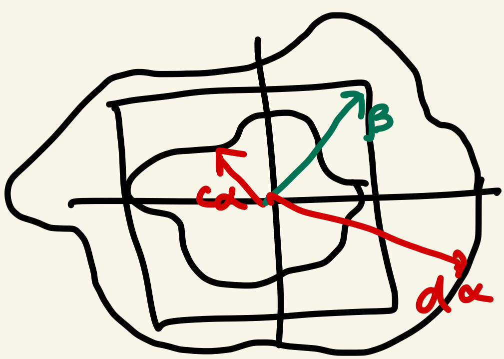
$$\|v\|_b = \beta.$$





$\|-\|_a$ $\|-\|_b$

The inequality $c\|v\|_a \le \|v\|_b \le d\|v\|_a$



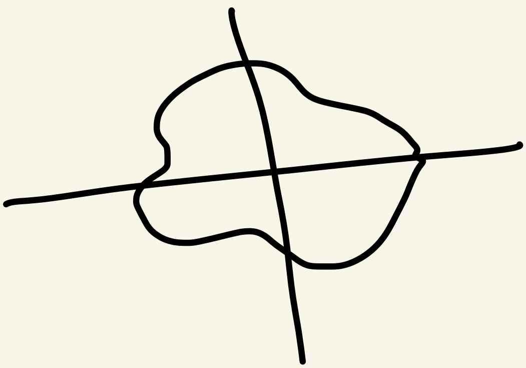
Pf Outline:

We need c, d s.t.

$$c\|v\|_a \leq \|v\|_b \leq d\|v\|_a$$

$\forall v.$

$$c = \min\{\|u\|_b \mid \|u\|_a = 1\}$$



$\|\cdot\|_b$ varies on the unit sphere of $\|\cdot\|_a$.

$$d = \max\{\|u\|_b \mid \|u\|_a = 1\}.$$

Inequality follows.
Revisit tomorrow

$\blacksquare$

Topology $\rightsquigarrow$ notion of an open set

$$\text{Fix } \tau \subseteq P(X)$$

$$\tau = \{\text{open sets of } X\}$$

$$\text{s.t. } \quad \phi \in \tau$$

$$X \in \tau$$

$$\text{If } U_i \in \tau \quad i \in I$$

$$\bigcup_{i \in I} U_i \in \tau$$

$$U_1 \cap \dots \cap U_n \in \tau.$$

$$\tau_1 \subseteq \tau_2.$$

$X = \mathbb{R}$ $\quad$ U is open if U^c is finite.