Recall: Equivalence of Norms

Given any two norms on $\mathbb{R}^n$, we have 2 unit spheres. (1 for each norm) and we sort of showed that you can stick each sphere inside of the other.

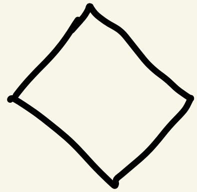
More explicity:

<u>Thm</u> Given $\|-\|_1$ and $\|-\|_2$ (not necessarily L^1 and L^2 norms), $\exists\, c, d \neq 0$ such that $\forall\, v \in \mathbb{R}^n$

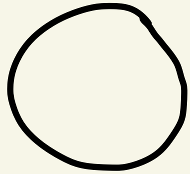
$$c\|v\|_1 \leq \|v\|_2 \leq d\|v\|_1$$

(c, d work $\forall v$)

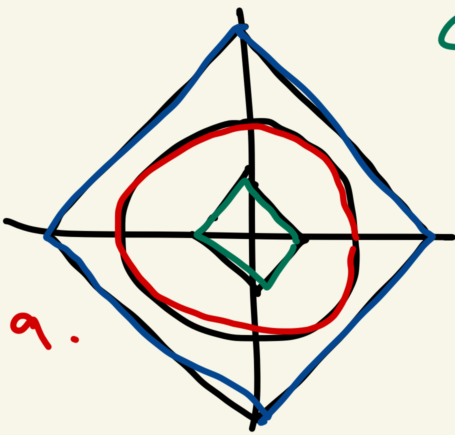
$\|-\|_1$



$\|-\|_2$



Thm says



$$\cancel{c\|v\|_1 \le \|v\|_2 \le d\|v\|_1}$$

This is backwards! See page 9.

$$c = \min\{\|u\|_2 \mid \|u\|_1 = 1\} \quad *$$

$$d = \max\{\|u\|_2 \mid \|u\|_1 = 1\} \quad *$$

Know this though.

Well for all unit vectors of $\|\cdot\|_1$

$$c \leq \|u\|_2 \leq d \quad \text{by definition}$$

but then given $v \in \mathbb{R}^n$,

$\dfrac{v}{\|v\|_1}$ is a unit vector $\|\cdot\|_1$.

$$c \leq \left\| \frac{v}{\|v\|_1} \right\|_2 \leq d$$

$$c \leq \frac{1}{\|v\|_1} \|v\|_2 \leq d$$

$$c\|v\|_1 \leq \|v\|_2 \leq d\|v\|_2. \quad \square$$

(The hard part is showing $c > 0$, $d < \infty$.)

Ex L^2-norm and L^∞-norm on $\mathbb{R}^n$.

$$c\,\|v\|_2 \leq \|v\|_\infty \leq d\,\|v\|_2.$$

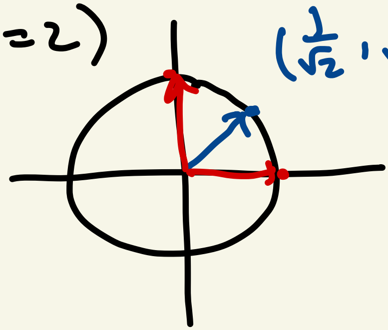
$$c\sqrt{v_1^2+\dots+v_n^2} \leq \max\{|v_1|,\dots,|v_n|\}$$
$$\leq d\sqrt{v_1^2+\dots+v_n^2}$$

$$c = \min\{\|u\|_\infty \mid \|u\|_2 = 1\}$$

$\|u\|_2 = 1 \qquad u_1^2+u_2^2+\dots+u_n^2 = 1$

Need to minimize the max of the u_i.

(n=2)



$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ has the minimum L^∞ norm.

In general

$$c = \min\left\{ \max\{|u_i|\} \mid u_1^2 + \dots + u_n^2 = 1 \right\}$$

$$= \frac{1}{\sqrt{n}}.$$

So $\vec{u} = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$

achieves the minimum.

$$d = \max\left\{ \|u\|_\infty \mid \|u\|_2 = 1 \right\}$$

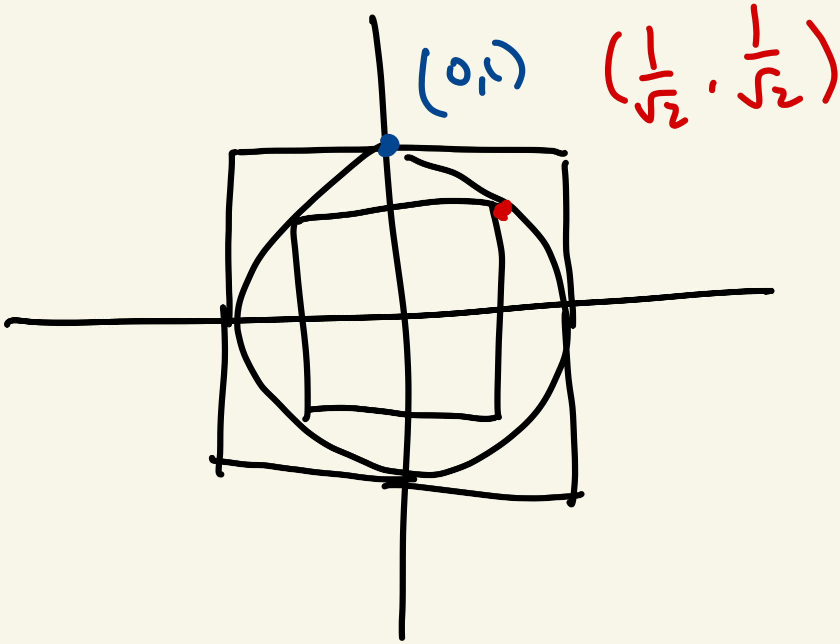
$u_i \leq 1$ and if we let

$$e_i = (0, 0, \dots, 1, \dots 0)$$

$\|e_i\| = 1.$

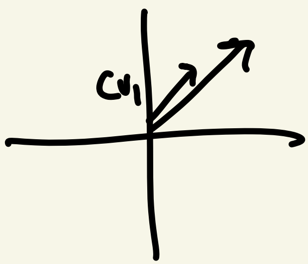
$\Longrightarrow$ $d = 1.$ Therefore, ...

$$\frac{1}{\sqrt{n}}\|v\|_2 \leq \|v\|_\infty \leq \|v\|_2 .$$



$$c\|v\|_1 \leq \|v\|_2$$

$$\|cv\|_1 \leq \|v\|_2$$



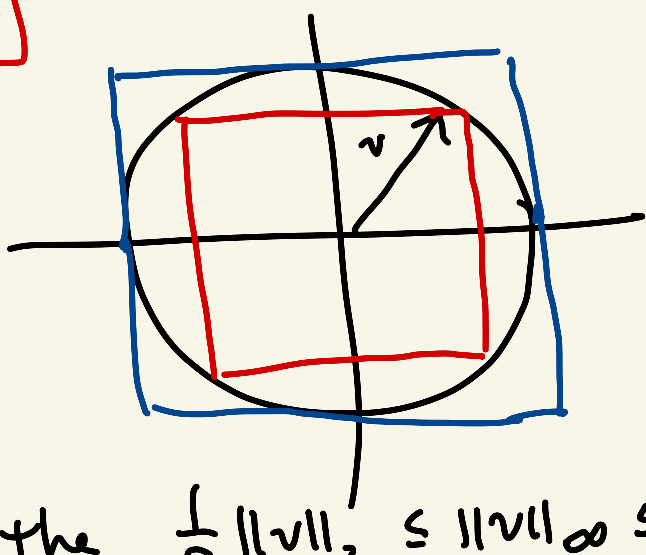
$$\frac{1}{\sqrt{n}}\sqrt{v_1^2+\dots+v_n^2}$$

$$=\sqrt{\frac{v_1^2}{n}+\dots+\frac{v_n^2}{n}} \leq \max\{|v_1|,\dots,|v_n|\}$$

$$\leq \sqrt{v_1^2+\dots+v_n^2}$$

Suppose $\|v\|_2 = 1$

$$\boxed{\frac{1}{\sqrt{2}}} \leq \|v\|_\infty \leq \boxed{1}$$



So the $\underline{\frac{1}{\sqrt{2}}\|v\|_2} \leq \|v\|_\infty \leq \underline{\|v\|_2}$.

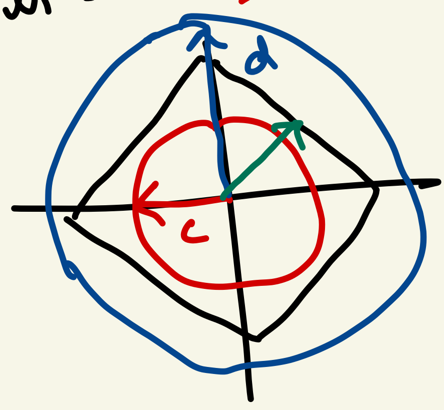
In general,

$$\boxed{c\|v\|_1} \le \|v\|_2 \le \boxed{d\|v\|_1}.$$

Given $\|v\|_1$ then what is $\|v\|_2$?

Suppose $\|v\|_1 = 1$ then what are bounds for $\|v\|_2$?

So it's $c \le \|v\|_2 \le d$.



Note: This only works in $\mathbb{R}^n$.

If you try to compare different norms, in say $C^0[a,b]$, then you won't get far.

Matrix Norms:

Recall: $M_{n\times n}(\mathbb{R})$ is also a vector space.

Given a norm $\|-\|$ on $\mathbb{R}^n$ we can define a norm on $M_{n\times n}(\mathbb{R})$ by

$$\|A\| = \max\{\|Au\| \mid \|u\| = 1\}$$

$\|A\| = \max\{\|Au\| \mid \|u\| = 1\}$.

- $\|A\| \geq 0$. Need to show if $\|A\| = 0 \Rightarrow A = 0$.

Let $\|A\| = 0$. Then

$$\max\{\|A\vec{u}\|\} = 0.$$

$\Rightarrow \|A\vec{u}\| = 0$ for all unit vectors.

But then if $\vec{v} \neq 0$, then

$\left\|A\frac{\vec{v}}{\|v\|}\right\| = 0$ (since $\frac{v}{\|v\|}$ is a unit)

$\Rightarrow \|Av\| = 0$ as well.

$\Rightarrow Av = 0$ $\forall v$ since $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

$\ker(A) = \mathbb{R}^n$, which means $A = 0$.

- $\|cA\| = \max\{\|cAu\|\}$

$$= \max\{|c|\,\|Au\|\}$$

$$= |c| \max\{\|Au\|\}$$

$$= |c|\,\|A\|$$

- $\|A+B\| = \max\{\|(A+B)u\|\}$

$$\leq \max\{\|Au\| + \|Bu\|\}$$

$$\leq \max\{\|Au\|\} + \max\{\|Bu\|\}$$

$$= \|A\| + \|B\|$$

$\square$

Take $\|\cdot\|_\infty$ on $\mathbb{R}^n$ and we'll define it on $M_{n\times n}(\mathbb{R})$.

$$\|A\|_\infty = \max\left\{ \|A\vec{u}\|_\infty \;\middle|\; \|u\|_\infty = 1 \right\}$$

Claim: $\|A\|_\infty$ = largest row sum

$$= \max\left\{ \sum_{j=1} |a_{ij}| \;\middle|\; i = 1, \dots, n \right\}$$

<u>Ex</u>

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 7 & -3 & 5 \end{bmatrix} = A$$

$$|-1| + |2| + |3| = 6$$

$$|5| + |-2| + |1| = 8$$

$$|7| + |-3| + |5| = 15$$

$$\|A\|_\infty = 15 = \max\{\|A\vec{u}\|_\infty\}$$

Pf :

$$\|A\|_\infty = \max\left\{ \boxed{\|Au\|_\infty} \;\middle|\; \|u\|_\infty = 1 \right\}$$

$$= \max\left\{ \max\{\text{entries of } Au\} \;\middle|\; \|u\| = 1 \right\}$$

$$= \max\left\{ \underbrace{\left|\sum a_{ij} u_j\right|}_{i^{\text{th}} \text{ entry of } Au} \;\middle|\; \|u\| = 1 \right\}$$

$$\leq \max\left\{ \sum |a_{ij} u_j| \;\middle|\; \|u\| = 1 \right\}$$

$$= \max\left\{ \sum |a_{ij}| \underbrace{|u_j|}_{\leq 1} \;\middle|\; \|u\|_\infty = 1 \right\}$$

$$\leq \max\left\{ \sum_{j=1}^{n} |a_{ij}| \;\middle|\; i = 1, \ldots, n \right\}$$

$$= \text{largest row sum}$$

Suppose row i achieves the largest row sum.

$$\sum_{j=1}^{n} |a_{ij}| \text{ is the largest row sum.}$$

Let $u \in S_1$ for the L^∞ norm.

defined by

$$u_j = 1 \text{ if } a_{ij} \geq 0$$

$$u_j = -1 \text{ if } a_{ij} < 0.$$

Ex $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ \underline{7} & \underline{-3} & \underline{5} \end{bmatrix}$

3 row is the biggest

$u = (\underline{1}, \underline{-1}, \underline{1})$

$\|A\|_\infty \geq \|Au\|_\infty$ for $u = (\pm 1, \pm 1, \ldots)$

in particular

But $\|A\vec{u}\|_\infty = \sum_{j=1}^{n} |a_{ij}|$ for the i^{th} row

= largest row sum.

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ \underline{7} & \underline{-3} & \underline{5} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ \boxed{15} \end{bmatrix}$$

$$\left((A)_{i*}\, u = \sum a_{ij} u_j = \sum |a_{ij}| \right)$$

$\|A\|_\infty \geq$ largest row sum

$\implies \|A\|_\infty =$ largest row sum

L^2-norm on $M_{2\times 2}(\mathbb{R})$ $(n=2)$

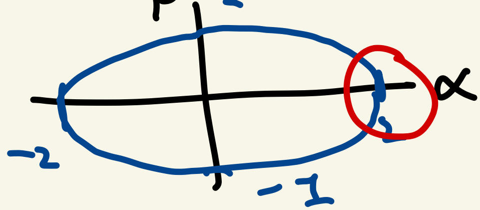
$$\left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\|_2$$

$$= \max \left\{ \left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 \;\middle|\; x^2 + y^2 = 1 \right\}$$

$$= \max \left\{ \left\| \begin{pmatrix} 2x \\ y \end{pmatrix} \right\|_2 \;\middle|\; x^2+y^2=1 \right\}$$

$$\begin{pmatrix} 2x \\ y \end{pmatrix} \;\Big|\; x^2+y^2=1$$

Let $\alpha = 2x$, $\beta = y$ $\Rightarrow \left(\frac{\alpha}{2}\right)^2 + \beta^2 = 1$

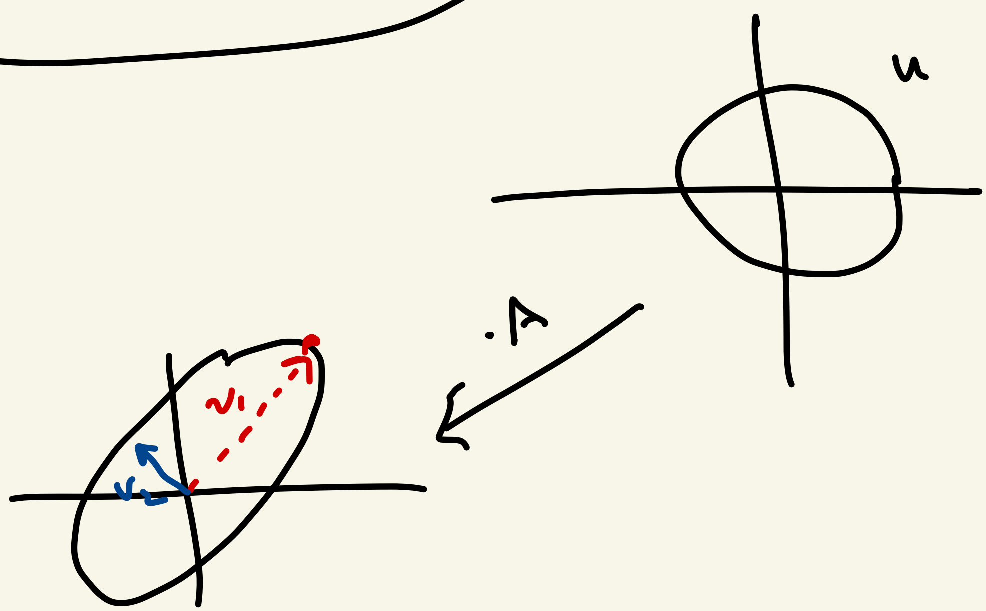


$$\left\| \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\|_2$$

$$= \max \left\{ \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|_2 \;\middle|\; \left(\frac{\alpha}{2}\right)^2 + \beta^2 = 1 \right\}$$

$$= \left\| \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\|_2$$

$$= 2$$



The norm $\|A\| = \max\{\|Au\|\}$

satisfies $\|AB\| \leq \|A\| \cdot \|B\|$.

$\Rightarrow$ define infinite series of matrices

$$\sum \frac{1}{n!} A^n = e^A$$ (diff eq topic)

§ 3.4 Positive Definite matrices

Back to inner products ...

let $\langle -,- \rangle$ ~~is~~ be an inner product on $\mathbb{R}^n$.

Recall the standard basis on $\mathbb{R}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\dots \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1 e_1 + x_2 e_2 + \dots + x_n e_n . \quad *$$

$$\langle x, y \rangle = \left\langle \sum_{i=1}^{n} x_i e_i , \sum_{j=1}^{n} y_i e_i \right\rangle$$

$$= \sum_{i=1}^{n} \left\langle x_i e_i , \sum_{j=1}^{n} y_j e_j \right\rangle$$

$$= \sum_{i=1}^{n} x_i \left\langle e_i , \sum_{j=1}^{n} y_j e_j \right\rangle$$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \langle e_i, y_j e_j \rangle$$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j \langle e_i, e_j \rangle$$

$$= \sum_{i,j=1}^{n} x_i y_j \overbrace{\langle e_i, e_j \rangle}^{\in \mathbb{R}}$$

Define $k_{ij} = \langle e_i, e_j \rangle$.

$$= \sum_{i,j=1}^{n} \boxed{k_{ij}} x_i y_j$$

what are these?

Every inner product has formula that is a linear combination of $x_i y_j$

(3.1.2d $x_1^2 y_1^2 + x_2^2 y_2^2$ not an inner product)

Define $K \in M_{n\times n}(\mathbb{R})$

$$(K)_{ij} = k_{ij} = \langle e_i, e_j \rangle$$

Ex: If $\langle x, y \rangle = \vec{x} \cdot \vec{y}$

then $e_1 \cdot e_1 = 1$ $\quad e_1 \cdot e_2 = 0$

$e_2 \cdot e_1 = 0$ $\quad e_2 \cdot e_2 = 1$

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3.1.2a $\langle x, y \rangle = 2x_1y_1 + 3x_2y_2$

$$K = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

(3.9) $\langle x, y \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2$

$$K = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

$\langle -, - \rangle$ on $\mathbb{R}^n$ $\rightsquigarrow$ matrix K

which matrices K $\rightsquigarrow$ inner products?

Claim: $\langle x, y \rangle = x^T K y$ (x^T is a row vector)

Pf: $x^T K y = (x_1 \dots x_n) K \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$= (x_1 \dots x_n) \left(\sum_{j=1}^{n} k_{ij} y_j \right)_{i=1,\dots,n}$$

$$= \sum_{i,j=1}^{n} k_{ij} x_i y_j = \langle x, y \rangle \quad \square$$

Any inner product is just $\underline{x^T K y}$, $x, y \in \mathbb{R}^n$.

Our question simplifies to which matrices K make an inner product of the form $\langle x, y \rangle = x^T K y$?

- Bilinear $x^T K y$ is bilinear no matter what K is

- Symmetry $y^T K x = x^T K y \quad \forall x, y$

If we let $x = e_i$ $y = e_j$

$$\begin{array}{ccc} e_i^T K e_j & = & e_j^T K e_i \\ \parallel & & \parallel \\ \langle e_i, e_j \rangle & & \langle e_j, e_i \rangle \\ \parallel & & \parallel \\ (K)_{ij} & & (K)_{ji} \end{array}$$

- Since $k_{ij} = k_{ji}$ $\Rightarrow$

  $K^T = K$ so that

  K is <u>symmetric</u>.

- Positivity

  $\langle x, x \rangle > 0$ if $x \neq 0$

  $\langle 0, 0 \rangle = 0$

  $0^T K 0 = 0$ no matter what K is.

  $\langle x, x \rangle = x^T K x > 0$ if $x \neq 0$.

Def Define the polynomial $q(x)$

$$= \langle x, x \rangle = \sum_{i,j}^{n} k_{ij}\, x_i x_j$$

$$K = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}.$$

positive definite.

$$\rightsquigarrow q(x_1, x_2)$$

$$= (x_1 \;\; x_2) \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= x_1^2 - 2x_1x_2 + 4x_2^2 > 0$$

$$= (x_1 - x_2)^2 + 3x_2^2 > 0$$

Def A symmetric matrix K is <u>positive definite</u> if

$$q(x) = x^T K x > 0 \quad \text{for all}$$

$$x \neq 0.$$

Every inner product $\langle -,- \rangle$ on $\mathbb{R}^n$ is of the form $\langle x,y \rangle = x^T K y$ for a positive definite matrix K. (1-1 correspondence ?)

We've narrowed down the study of inner products to positive def. matrices.

Note: Positive def has no relation to the entries being positive.

$\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$ was pos. def. despite having negative entries.

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not pos def. despite having all positive entries.

Gram matrices:

Let V be an inner product space.

Let $v_1, \ldots, v_n \in V$. The Gram matrix for the vectors is the $n \times n$ matrix K

$$\text{w/} \quad (K)_{ij} = \langle v_i, v_j \rangle .$$

Def: A matrix K is positive semi-definite if

$$q(x) = x^T K x \geq 0 .$$

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is positive semi-def. but not positive-def.

Thm Let V be an inner product space. Let K be the Gram matrix for the set $v_1, \ldots v_n \in V$.

Then K is positive semi-definite. $*$

Furthermore, K is positive definite iff $v_1, \ldots, v_n$ are independent.

If $v_1, \ldots, v_n \in V$ all you have to do is check whether

$$K = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \\ \vdots & \ddots & \\ \vdots & & \langle v_n, v_n \rangle \end{pmatrix}$$

is positive def or not, to see if the v_i are independent.

Pf in book

Ex: let $V = C^0[-\pi, \pi]$
" $=$ " $C^0[0, 2\pi]$

let $f(x) = 5$
$g(x) = 2\sin^2(x) - 1$
$h(x) = 3\cos^2(x)$ (dependent)

But we can check using Gram matrix.

$$K = \begin{pmatrix} \langle f,f \rangle & \langle f,g \rangle & \langle f,h \rangle \\ \langle g,f \rangle & \langle g,g \rangle & \langle g,h \rangle \\ \langle h,f \rangle & \langle h,g \rangle & \langle h,h \rangle \end{pmatrix}$$

$$\langle f,g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

$$K = \begin{pmatrix} 50\pi & 0 & 15\pi \\ 0 & \pi & -\frac{3}{2}\pi \\ 15\pi & -\frac{3}{2}\pi & \frac{27\pi}{4} \end{pmatrix}$$

$$= \frac{\pi}{4}\begin{pmatrix} 200 & 0 & 60 \\ 0 & 4 & -6 \\ 60 & -6 & 27 \end{pmatrix}$$

Is $\begin{pmatrix} 200 & 0 & 60 \\ 0 & 4 & -6 \\ 60 & -6 & 27 \end{pmatrix}$ positive definite?

Prop Any positive def matrix is invertible.

Pf let $z \in \ker(K)$

$$\Rightarrow \quad Kz = 0$$

$$z^T K z = 0$$

$$\Rightarrow \quad z = 0. \quad \text{So } \ker(K) = 0$$

$$\Longleftrightarrow \quad K \text{ is invertible.}$$

$\begin{pmatrix} 200 & 0 & 60 \\ 0 & 4 & -6 \\ 60 & -6 & 27 \end{pmatrix}$ not invertible!

$\Longrightarrow$ not pos def.

$z = \begin{pmatrix} -3 \\ 15 \\ 10 \end{pmatrix} \Rightarrow z^T K z = 0.$

f, s, h are dependent!