HW 5: Slight change

3.3.35 only do part i).

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§ 3.6 Complex Vector Spaces

Complex numbers

$\mathbb{C}$ is the set of complex numbers

$$= \{ a + ib \mid a, b \in \mathbb{R},\ i^2 = -1 \}.$$

$i \notin \mathbb{R}$. If you try to solve

$x^2 + 1 = 0$, you won't get a

sol'n in $\mathbb{R}$. So you can

"add" one "i", and make $\mathbb{C}$.

If you have a complex polynomial,

(it's a polynomial w/ complex coefficients,

e.g. $(3+i)x^2 + (2-i)x + (3+4i)$)

it will always complex solutions.

$x^2+x+1=0 \longrightarrow$ complex sol'ns.

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$\mathbb{C}$ is called "algebraically closed"

since it contains all solutions to it's

polynomials.

Let $z = a + bi$.

Define $\bar{z} = a - bi$, the complex conjugate of z.

The product $z\bar{z} \in \mathbb{R}$.

$$z\bar{z} = (a+bi)(a-bi) = a^2 + abi - abi + b^2$$

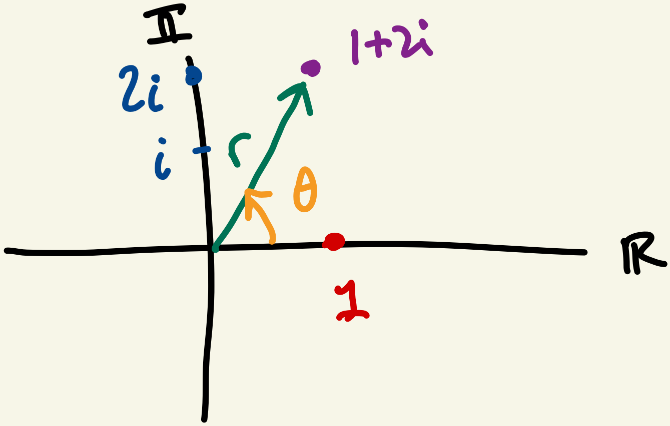
$$= a^2 + b^2 \in \mathbb{R}.$$

<u>Def</u> The absolute value of $|a+bi| = \sqrt{a^2+b^2}$.

Then $z\bar{z} = |z|^2$.

(Define $|z| = \sqrt{z\bar{z}} \in \mathbb{R}$)

Compare this $\|v\|_2 = \sqrt{v \cdot v}$ over $\mathbb{R}$



$x + yi$

Cartesian

Radial $\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$r = \sqrt{x^2+y^2} = |z|$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

??

$1+2i \longrightarrow \sqrt{1^2+2^2}\,\boxed{e^{i\theta}}$

$\sqrt{5}\; e^{i\tan^{-1}\left(\frac{2}{1}\right)}$

$\cos\theta = \frac{x}{r}$ $\quad \sin\theta = \frac{y}{r}$ $\quad z = x + iy$

$r = |z|$

$x = |z|\cos\theta$

$y = |z|\sin\theta$

$$z = |z|\cos\theta + i|z|\sin\theta$$

$$= |z|(\cos\theta + i\sin\theta)$$

$$= \boxed{r}(\cos\theta + i\sin\theta)$$

$r = \sqrt{x^2+y^2} = z\bar{z}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$= re^{i\theta} \; *$$

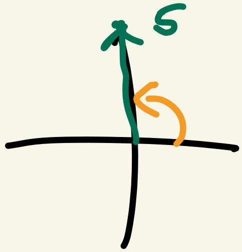
since $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Pf using inf series

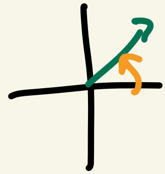
Def $e^{i\theta} = \sum_{n=1}^{\infty} \frac{1}{n!}(i\theta)^n$

$\Rightarrow \cos\theta = \frac{e^{-i\theta} + e^{i\theta}}{2}$ etc

$z = 5e^{i\frac{\pi}{2}} = 5i$



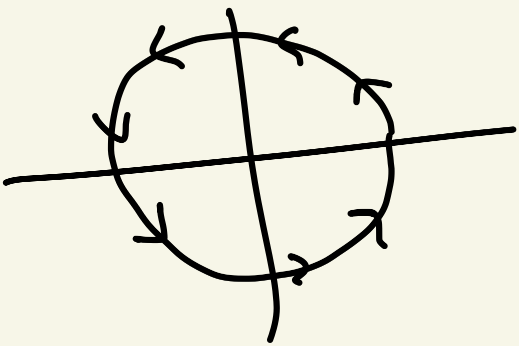
$z = 1e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$



If θ is a function of time $\underline{\theta(t)}$

then

$$p(t) = e^{i\theta(t)} \quad (r = 1)$$



Complex vector spaces

$\mathbb{C}$ is just as valid as a set of scalars as $\mathbb{R}$.

Gaussian Elimation

$r_i' = cr_i + r_j$

c could be complex no problem

$r_i' = r_j \quad r_j' = r_i$

nothing happens

$r_i' = cr_i$.

complex numbers ✓

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix}$$

→ should be a leading 1

$r_3' = \frac{1}{5} r_3$

You definitely need to be able to divide scalars to do Gaussian elimination

Scalars should have the following abilities ...

$+, -, \times, \div$

distribute property, associ., comm.

The complex numbers can do all these things just as good as $\mathbb{R}$.

<u>Def</u> A complex vector space is a set V s.t. $v + w \in V$ and $cv \in V$ w/ $c \in \mathbb{C}$ s.t. addition and scalar mult. satisfy the 7 axioms from before.

Ch 1 and 2 are exactly the same.

Ex $V = \mathbb{C}^n$

e.g $\begin{pmatrix} 5-i \\ 2 \\ 10+20i \end{pmatrix} \in \mathbb{C}^3$

($\mathbb{R}^n$ is almost a subspace of $\mathbb{C}^n$).

$(5+i)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \notin \mathbb{R}^3$.

Ex $V = C^0[a,b]$

$= \{ f(x) = u(x) + i\,v(x) \}$

$f: \mathbb{R} \to \mathbb{C}$

= "parametrization of a curve in $\mathbb{C}$"

$\mathbb{C}^3$ can be viewed as a 3-dim complex v.s w/ basis vectors

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$\mathbb{C}^3$ is also a real vector space that's 6 dimensional.

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}i\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\i\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\0\\i\end{pmatrix}.$$

If you mention a vector space, be specific about scalars.

Dot product on $\mathbb{C}^n / \mathbb{C}$

Guess: $\vec{z}, \vec{w} \in \mathbb{C}^n$

$$\vec{z} \cdot \vec{w} = z^T w$$

$$= (z_1 \; \cdots \; z_n) \begin{pmatrix} v_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} z_i w_i$$

We want to preserve the fact

$$\|\vec{z}\| = \sqrt{\langle z, z \rangle} \in \mathbb{R}$$

Need $\langle z, z \rangle \in \mathbb{R}$.

$$z = \begin{pmatrix} 1-i \\ 2 \end{pmatrix} \in \mathbb{C}^2.$$

$$\|z\|^2 = z \cdot z = (1-i)^2 + 2^2$$

$$= 1 - 2i - 1 + 4 = 4 - 2i \notin \mathbb{R}$$

Actual dot product on $\mathbb{C}^n/\mathbb{C}$.

$|z|^2 = z\bar{z} \in \mathbb{R}$.

complex scalars

<u>Def</u> let $V = \mathbb{C}^n$ over $\mathbb{C}$.

$\mathbb{C}^n/\mathbb{C}$

$(\neq \mathbb{C}^n/\mathbb{R})$

$$\vec{z} \cdot \vec{w} = z^T \bar{w}$$

$$= (z_1 \dots z_n)\begin{pmatrix} \bar{w}_1 \\ \vdots \\ \bar{w}_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} z_i \bar{w}_i$$

<u>Ex</u> $(1-i, 2) \cdot (1+i, -i)$

$$= (1-i)(\overline{1+i}) + 2\,\overline{(-i)}$$

$$= (1-i)^2 + 2i$$

$$= 1 - 2i - 1 + 2i$$

$$= 0$$

No longer is $\underset{\bar{w}}{\underline{z \cdot w}} = \underset{\bar{z}}{w \cdot z}$

Turns out that $z \cdot w = \overline{w \cdot z}$

Furthermore $z \cdot z = z^T \bar{z}$

$$= (z_1 \ldots z_n) \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} z_i \bar{z_i} = \sum_{i=1}^{n} |z_i|^2 \in \mathbb{R}$$

So define $\|\vec{z}\| = \sqrt{z \cdot z} \in \mathbb{R}$.

$$\left\| \begin{pmatrix} 1-i \\ 2 \end{pmatrix} \right\| = \sqrt{(1-i)(1+i) + 2 \cdot 2}$$

$$= \sqrt{2+4} = \sqrt{6}$$

Complex Inner product Space

An inner product on a complex vector space $V/\mathbb{C}$

is a pairing $\langle -,- \rangle : V \times V \to \mathbb{C}$

s.t.

- $\langle cu + dv, w \rangle = c\langle u,w \rangle + d\langle v,w \rangle$
- $\langle u, cv + dw \rangle = \bar{c}\langle u,v \rangle + \bar{d}\langle u,w \rangle$
- $\langle v,w \rangle = \overline{\langle w,v \rangle}$
- $\langle v,v \rangle := \|v\|^2 \geq 0$
  w/ $\|v\| = 0$ iff $v = 0$.

Notice that $\bar{x} = x$, so these axioms become the original axioms if you change $\mathbb{C}$ back to $\mathbb{R}$.

3.3.35i 3.3.39

Show the equivalence of the Euclidean norm and the 1 norm

$\alpha = \sqrt{n}$

$$\underbrace{\|v\|_2 \le \|v\|_1} \le \sqrt{n}\,\|v\|_2$$

$\|v\|_1 = |v_1| + \dots + |v_n|$

$\|v\|_1^2 = (|v_1| + \dots + |v_n|)^2$

$= |v_1|^2 + |v_2|^2 + \dots + |v_n|^2$

$+$ positive stuff

$\ge |v_1|^2 + \dots + |v_n|^2$

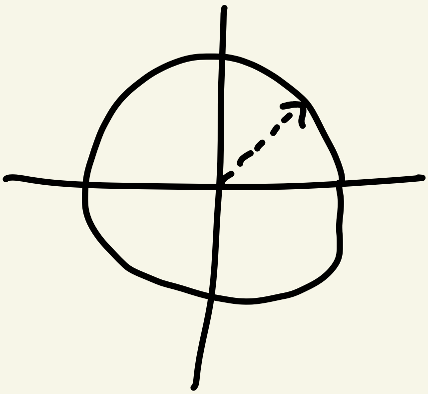
$= \|v\|_2^2$

$$\|v\|_1^2 \leq n\|v\|_2^2$$

$$n\left(v_1^2 + \dots + v_n^2\right)$$

$$\geq \sum_{i,j=1}^{n} |v_i||v_j|$$

$$d = \max\left\{ \|u\|_1 \;\middle|\; \|u\|_2 = 1 \right\}$$



$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

In general $\quad d = \max\left\{ \|u\|_1 \;\middle|\; \|u\|_2 = 1 \right\}$

$$= \sum_{i=1}^{n} \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$$

You need to find the unit vector on unit ball of the ℓ^2-norm w/ maximal ℓ^1-norm.

Claim: $u = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$.

is the maximal size by

$$\|v\|_1 \leq \sqrt{n}\,\|v\|_2$$

$$\left\| \frac{v}{\|v\|_2} \right\|_1 \leq \sqrt{n}$$

$$\|v\|_1 \leq \sqrt{n}\,\|v\|_2$$

$$\sqrt{n}\sqrt{v_1^2 + \dots + v_n^2}$$

$$\overset{?}{=} |v_1| + \dots + |v_n|$$

$$n(v_1^2 + \dots + v_n^2) \geq (|v_1| + \dots + |v_n|)^2$$

$$\left(\frac{a_1 + \dots + a_n}{n}\right)^2 \leq \frac{(a_1)^2 + \dots + (a_n)^2}{n}$$

$$\frac{a_1^2 + \dots + a_n^2}{n^2} + \text{junk}$$

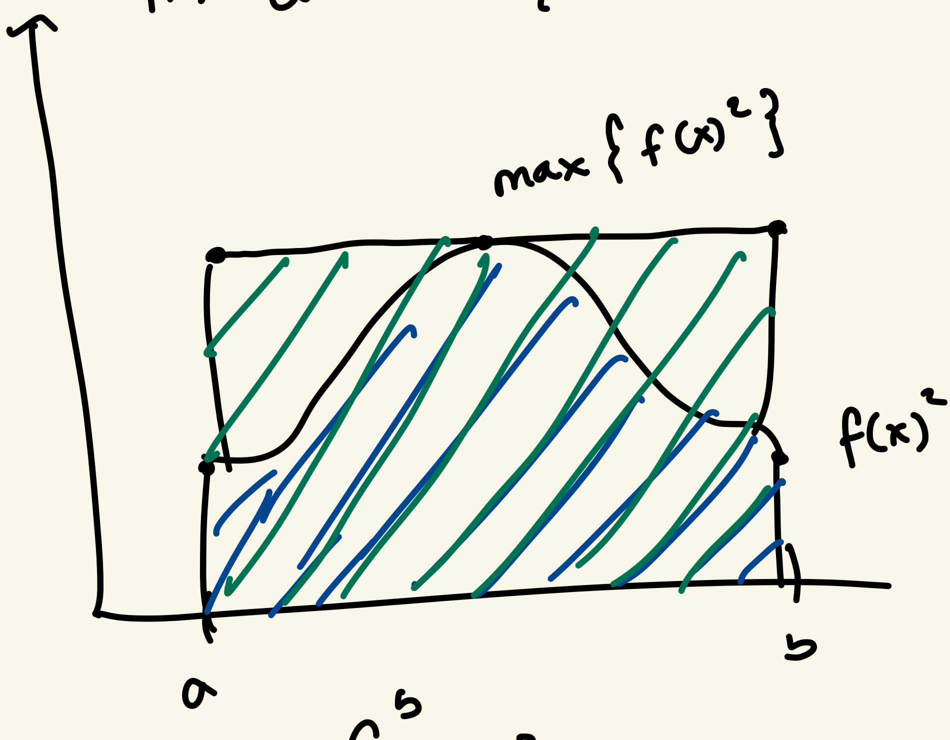
$$\leq \frac{a_1^2 + \dots + a_n^2}{n}$$

$$\|v\|_1 = (|v_1|, \dots, |v_n|) \cdot (1, 1, \dots, 1)$$

$$\leq \|(|v_1|, \dots, |v_n|)\| \cdot \|(1, 1, \dots, 1)\|$$

$$= \|v\|_2 \cdot \sqrt{n}$$

$$\|f\|_\infty = \max \{ |f(x)| \mid x \in [a,b] \}$$



$$\int_a^b f(x)^2 \, dx \leq \int_a^b \|f\|_\infty^2 \, dx = (b-a) \|f\|_\infty^2$$

$$\int \boxed{f(x)^2} \leq \max\{f(x)^2 \mid x \in [a,b]\}$$

$$= \max\{|f(x)| \mid x \in [a,b]\}^2$$

$\|f\|_2^2$

$$= \int \boxed{\|f\|_\infty^2} = \text{constant function}$$

$$\underline{\int_a^b f(x)^2\,dx \leq \int_a^b \|f\|_\infty^2\,dx}$$

$$= \left(\|f\|_\infty^2 x\right)_a^b$$

$$= \|f\|_\infty^2 (b-a)$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left(\frac{\langle v, w\rangle}{\|v\|\,\|w\|}\right)^2$$

$$= 1 - \left(\frac{v_1w_1 + v_2w_2}{\sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}}\right)^2$$

$$= 1 - \frac{(v_1w_1 + v_2w_2)^2}{(v_1^2 + v_2^2)(w_1^2 + w_2^2)}$$

$$= \frac{\|v\|\,\|w\|}{\|v\|\,\|w\|} - \frac{(v_1w_1)^2 + 2v_1w_1v_2w_2 + (v_2w_2)^2}{(}$$

$$= \frac{\|v\|^2\|w\|^2}{\|w\|^2\|w\|^2} - \frac{\langle v, w\rangle^2}{\|v\|^2\|w\|^2}$$

$$= \frac{\|v\|^2\|w\|^2 - \langle v,w\rangle^2}{\|v\|^2\|w\|^2} \qquad 5.2.10b$$

$$\|v\|^2\|w\|^2 \sin^2\theta = \|v\|^2\|w\|^2 - \langle v,w\rangle^2$$

$$\|v\|^2\|w\|^2 - \langle v,w\rangle^2$$

$$= (v_1^2+v_2^2)(w_1^2+w_2^2) - (v_1w_1+v_2w_2)^2$$

$$= \cancel{(v_1)^2 w_1^2} + v_2^2w_1^2 + v_1^2w_2^2 + \cancel{v_2^2w_2^2}$$

$$- \cancel{v_1^2w_1^2} - 2v_1w_1v_2w_2 + \cancel{v_2^2w_2^2}$$

$$= v_2^2w_1^2 + v_1^2w_2 - 2v_1w_1v_2w_2$$

$$= \underline{(v_1w_2 - v_2w_1)^2}$$

$$\sin^2\theta \; \|v\|^2 \, \|w\|^2 = \underbrace{(v_1w_2 - v_2w_1)^2}$$

If $\theta \in [0, \pi]$

$$(v \times w)^2 = \|v\|^2 \, \|w\|^2 - \langle v, w \rangle^2$$

$$= \|v\|^2 \, \|w\|^2 \sin^2\theta$$

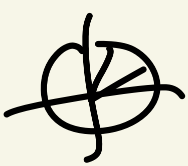
?

$$\pm \boxed{v \times w} = \underbrace{\|v\| \, \|w\| \sin\theta}$$

$\sin\theta = 0$ when v, w parallel

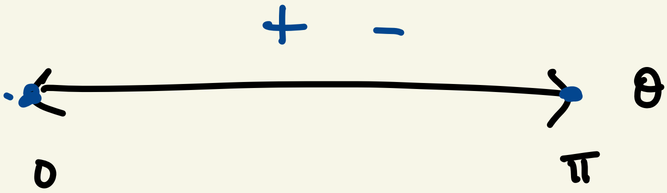
$$v_1w_2 - v_2w_1 \geq 0$$

$$v_1w_2 \geq v_2w_1$$



$$\frac{v_1}{v_2} \geq \frac{w_1}{w_2} \qquad \frac{1}{\tan\theta_1} \geq \frac{1}{\tan\theta_2}$$

$$\tan\theta_2 \geq \tan\theta_1$$



Fix $\underline{\|v\| \, \|w\|}$
and
vary θ

Pick $\theta = 90° = \pi/2$ to test
whether $\boxed{v \times w}$ is + or − .

$$(v_1, v_2) \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{\|w\|}{\|v\|} \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} \Big\}$$

$$(v_1, v_2) \times (-v_2 \; v_1)$$

$$v_1^2 - (-v_2)(v_1) = v_1^2 + v_2^2 > 0$$

OR $v = (1, 0)$ $\quad w = (0, \|w\|)$

$$v \times w > 0 .$$

3.3.28

$\|-\|_2$ on $\mathbb{R}^2$

$$(-5,2) \rightsquigarrow \frac{1}{\|(-5,2)\|_2}(-5,2)$$

is a unit vector

$$= \frac{(-5,2)}{\sqrt{29}}$$

(a) L^1-norm on $C^0[0,1]$

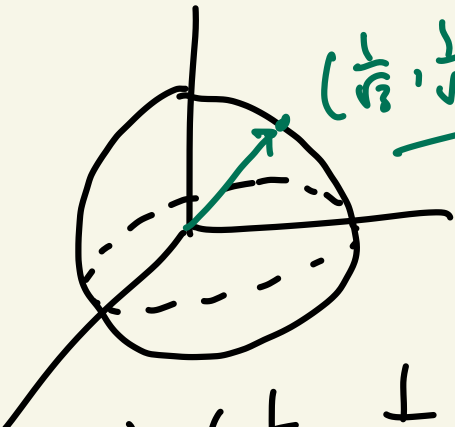
unit vector for $f(x) = x - \frac{1}{3}$.

$$f \longrightarrow \frac{1}{\|f\|_1} f \quad \text{is a unit vector}$$

$$\frac{1}{\int_0^1 |f(x)|\,dx}\left(x - \frac{1}{3}\right) = \text{something}$$

$$c = \min\{\|u\|_1 \mid \|u\|_2 = 1\} = 1$$

$$d = \max\{\|u\|_1 \mid \|u\|_2 = 1\} = \sqrt{n}$$



$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

this point has maximum L^1-norm out of all unit vectors in the L^2 norm.

$$\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\|$$

$$= \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3} = d$$

$v+w = (v_1+w_1, \; v_2+w_2)$

$\|v+w\|^2$

$= 2(v_1+w_1)^2 - (v_1+w_1)(v_2+w_2)$

$+ 2(v_2+w_2)^2$

$= 2\,(v_1^2 + 2v_1w_1 + w_1^2)$

$-($

$\langle v, w\rangle = 2v_1w_1 - \frac{1}{2}v_1w_2$

$- \frac{1}{2}v_2w_1 + 2v_2w_2$

is an inner product

also that

$\|\langle v,v\rangle\|^2 = 2v_1^2 - v_1v_2 + 2v_2^2$

so it satisfies

Δ - ineq.

Start with

$$(\|v\| + \|w\|)^2$$

$$= \|v+w\|^2 + \text{extra positive terms}$$

$$\geq \|v+w\|^2$$

(c) $\|v+w\| = 2|v_1+w_1| + |v_2+w_2|$

$$\leq 2(|v_1| + |w_1|) + |v_2| + |w_2|$$

$$= (2|v_1| + |v_2|) + (2|w_1| + |w_2|)$$

$$= \|v\| + \|w\|$$

(a) $\|v\| = \max\{2|v_1|, |v_2|\}$

$$\|(1,3)\| = \max\{2|1|, |3|\}$$

$$= \max\{2, 3\} = 3$$

$$\|(-5,1)\| = \max\{2\cdot|-5|, |1|\}$$

$$= \max\{10, 1\} = 10$$

$$\|v+w\| = \max\{2|v_1+w_1|, |v_2+w_2|\}$$

$$\leq \max\{2(|v_1| + |w_1|), |v_2| + |w_2|\}$$

$$= \max\{2\underline{|v_1|} + 2|v_1|, \underline{|v_2|} + |w_2|\}$$

$$\leq \max\{2|v_1|, |v_2|\} + \max\{2|w_1|, |w_2|\}$$

$$\|v\| + \|w\|$$