Quick Recap

Complex vector spaces are vector spaces but with complex scalars.

All the results from Ch 1 and 2 are the same for complex vector spaces.

But complex inner products are slightly different.

- $\langle cu + dv, w \rangle = c\langle u, w\rangle + d\langle v, w\rangle$
- $\langle u, cv + dw \rangle = \bar{c}\langle u, v\rangle + \bar{d}\langle u, w\rangle$

$\ne$

$\overline{(x+iy)} = x - iy$

- $\langle v, w \rangle = \overline{\langle w, v \rangle}$ $*$

- $\langle v, v \rangle > 0 \quad \forall v \neq 0$ and

  $\langle 0, 0 \rangle = 0.$

If $\langle v, v \rangle$ had an imaginary component, then we would lose positivity axiom and $\|v\|^2 = \langle v, v \rangle$ wouldn't be a real number even.

<u>Ex</u> $\underbrace{C^0[-\pi, \pi] / \mathbb{C}}$ scalars $= \mathbb{C}$
complex v.s.

Define $C^0[-\pi, \pi] = \{ f : [-\pi, \pi] \to \mathbb{C} \}$

$f(x) = u(x) + i v(x)$
is the form of these functions

$f(x) = 2x + i(5x)$ for example

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Let k be an integer

$$k \in \mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$$

$$f_k(x) = e^{ikx} = \cos(kx) + i\sin(kx)$$

$$f_k(x) \in C^0[-\pi, \pi]$$

This vs has inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx$$

(Think $\langle v, w \rangle = v^T \overline{w}$)

Ex $\langle e^{ikx}, e^{ilx} \rangle$ $\quad k, l$ are integers

$$= \int_{-\pi}^{\pi} e^{ikx} \cdot \overline{e^{ilx}} \, dx$$

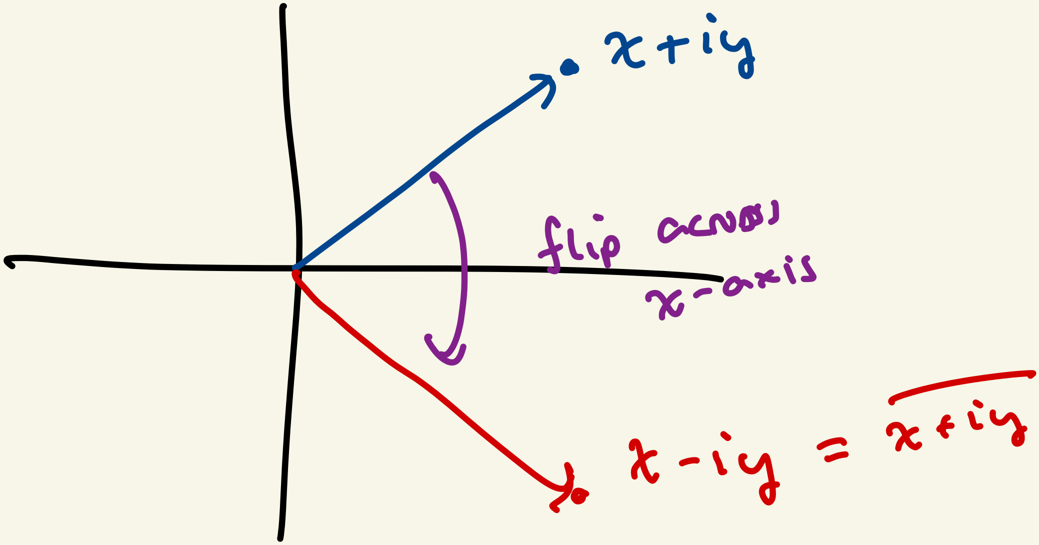
Side: $\overline{e^{ilx}} = \overline{\cos(lx) + i\sin(lx)}$

$$= \cos(lx) - i\sin(lx)$$

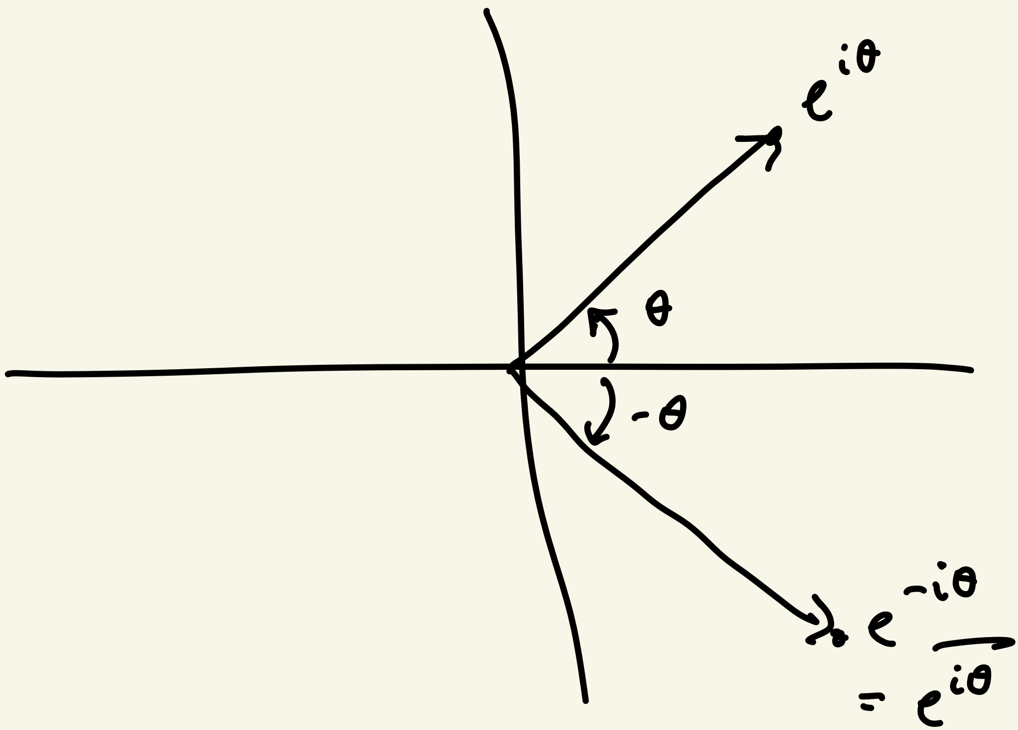
$$= \cos(-lx) + i\sin(-lx)$$

$$= e^{-ilx}$$

(In general, $\overline{e^{i\theta}} = e^{-i\theta}$)



$x+iy$
flip across x-axis
$x-iy = \overline{x+iy}$



$e^{i\theta}$
θ
$-\theta$
$e^{-i\theta}$
$= \overline{e^{i\theta}}$

$$\langle e^{ikx}, e^{ilx} \rangle = \int_{-\pi}^{\pi} e^{ikx} \overline{e^{ilx}} \, dx$$

$$= \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} \, dx$$

$$= \int_{-\pi}^{\pi} e^{i(k-l)x} \, dx$$

2 cases

Case 1: $k = l$

$$= \int_{-\pi}^{\pi} e^{i(0)x} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi$$

$$\| e^{ikx} \|^2 = \int_{-\pi}^{\pi} e^{ikx} \cdot \overline{e^{ikx}} \, dx = 2\pi$$

Case 2 : $k \neq l$

$$= \int_{-\pi}^{\pi} e^{i(k-l)x} \, dx \qquad u = i(k-l)x, \quad dx = \frac{1}{i(k-l)} du$$

$$= \left(\frac{e^{i(k-l)x}}{i(k-l)} \right)_{-\pi}^{\pi} \quad *$$

optional

$$= \left(\frac{1}{i(k-l)} \Big(\cos((k-l)x) + i \sin((k-l)x) \Big) \right)_{-\pi}^{\pi}$$

$$= \frac{1}{i(k-l)} \Big(\cancel{\cos((k-l)\pi)} + i \sin((k-l)\pi) - \cos(-(k-l)\pi) - i \sin(-(k-l)\pi) \Big)$$

Since $k-l \in \mathbb{Z}$

$(k-l)\pi$ and $(k-l)(-\pi)$ are the same angle

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$-\pi, \pi$ same

$-2\pi, 2\pi$ same angle ($k-l$ is an integer)

$\vdots$

$$\frac{1}{i(k-l)}\left(e^{\overbrace{i(k-l)\pi}} - e^{\overbrace{i(k-l)(-\pi)}} \right)$$

$$\langle e^{ikx}, e^{ilx} \rangle = 0$$

And so e^{ikx} and e^{ilx} are orthogonal complex functions when $k \neq l$.

§ 4.1 Orthogonal Bases

<u>Def</u> Let V be an inner product space.

Let $v_1, \ldots, v_k \in V$. ($\mathbb{R}$ or $\mathbb{C}$)

We say that $v_1, \ldots, v_k$ are <u>mutually orthogonal</u> when

$\langle v_i, v_j \rangle = 0 \quad \forall\, i \neq j$.

(Mostly $V/\mathbb{R}$ but this works fine over $\mathbb{C}$.)

<u>Ex</u> $e_1, e_2, \ldots, e_n$ are mutually orthogonal in $\mathbb{R}^n$ w/ dot product.

$$\langle e_i, e_j \rangle = e_i \cdot e_j$$

$$= 0 \cdot 0 + \cdots + \underset{i}{1} \cdot 0 + \cdots + \underset{j}{0} \cdot 1 + \cdots$$

$\langle e_i, e_j \rangle = 0$ for all $i \neq j$.

<u>Ex</u> e^{ikx} $(k \in \mathbb{Z})$ are mutually orthogonal in $C^0[-\pi,\pi]/\mathbb{C}$

$$\langle e^{ikx}, e^{ilx} \rangle = 0 \quad \text{for } (k \neq l)$$

<u>Def</u> Let $\{v_1, \dots v_n\} \subseteq V$, an inner product space. We say that $\{v_1 \dots v_n\}$ form an <u>orthogonal</u> basis if they are a basis and mutually orthogonal.

<u>Ex</u> $\{e_1, \ldots, e_n\}$ form an orthogonal basis of $\mathbb{R}^n$.

<u>Non Ex</u> $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^2$ w/ dot product.

But it's not an orthogonal basis.

<u>Ex</u>

But if

$$\langle (v_1, v_2), (w_1, w_2) \rangle$$

$$= v_1 w_1 - v_1 w_2 - v_2 w_1 + 4 v_2 w_2$$

$$\langle (1,0), (1,1) \rangle = 1 \cdot 1 - 1 \cdot 1 - 0 \cdot 1 + 4 \cdot 0 \cdot 1$$

$$= 0$$

So it is an orthogonal basis on $\mathbb{R}^2$ w/ this weighted inner product

Def We say a basis $\{u_1, \ldots, u_n\}$ of V is __orthonormal__ if it is orthogonal and

$\|u_i\| = 1$.

Ex Any orthogonal basis can be turned into an orthonormal basis

$$\{v_1, \ldots, v_n\} \longrightarrow \left\{\frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|}\right\}$$

Ex $\{e_1, \ldots, e_n\}$ are an orthonormal basis of $\mathbb{R}^n$ w/ dot product

Prop Let $\{u_1, \dots, u_n\}$ be an orthonormal basis of an inner product space V. ($/\mathbb{R}$)

Then $\forall v \in V$

$$v = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_n \rangle u_n$$

and $$\|v\|^2 = \sum_{i=1}^{n} \langle v, u_i \rangle^2$$

looks like the usual dot product norm formula.

Pf Since $\{u_1, \dots, u_n\}$ is a basis, then

$$v = c_1 u_1 + \dots + c_n u_n.$$

We need to show that $c_i = \langle v, u_i \rangle$. $\forall i$.

just compute this inner product

$$\boxed{\langle v, u_i \rangle}$$

$$= \left\langle \sum_{j=1}^{n} c_j u_j , u_i \right\rangle$$

$$= \sum_{j=1}^{n} c_j \langle u_j, u_i \rangle \quad \text{(bilinearity)}$$

If $j \neq i$, then $\langle u_j, u_i \rangle = 0$

If $j = i$, then $\langle u_i, u_i \rangle = 1$.

since u_i is a unit vector

$$= c_i \langle u_i, u_i \rangle = \boxed{c_i}$$

So $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

$$\|v\|^2 = \langle v, v \rangle$$

$$= \left\langle \sum_i c_i u_i , \sum_i c_i u_i \right\rangle$$

{here V/ℝ is used}

$$= \sum_{i,j=1} c_i c_j \langle u_i , u_j \rangle$$

Since $\langle u_i , u_j \rangle = 0$ when $i \neq j$

$$= \sum_{i=j=1}^{n} c_i^2 \langle u_i , u_i \rangle$$

$$= \sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} \langle v, u_i \rangle^2$$