Recall An orthonormal basis of an inner product space V is a basis $\{u_1, \ldots, u_n\}$ s.t.

$$\langle u_i, u_j \rangle = 0 \quad \forall\, i \neq j$$

and $\|u_i\| = 1$.

If we get rid of the unit vector criterion, we get an orthogonal basis.

Thm If $v = c_1 u_1 + \ldots + c_n u_n$ where $\{u_1 \ldots u_n\}$ orthonormal,

then

$$\|v\|^2 = c_1^2 + c_2^2 + \ldots + c_n^2. \Big\}$$

No matter what $\|-\|^2 = \langle -, - \rangle$ is!

So an inn prod space w/ an orthonormal basis essentially is like $\mathbb{R}^n$ w/ dot product and standard basis.

You can compute the coefficients w/ this formula.

$c_i = \langle v, u_i \rangle$

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i \quad \forall v \in V.$$

You can compute a linear combination w/out row reduction.

$$\begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = v$$

$\xrightarrow{\text{instead}}$ $c_i = \langle v, u_i \rangle$

Instead what if our basis $v_1, v_2, \ldots, v_n$ is <u>orthogonal</u>?

Then $\forall\, v \in V$

$$v = c_1 v_1 + \ldots + c_n v_n$$

where $c_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$.

<u>Quick Pf</u>

$$\langle v, v_i \rangle = \sum c_j \langle v_j, v_i \rangle = c_i \langle v_i, v_i \rangle$$

$$= c_i \|v_i\|^2$$

$\left(\langle v_j, v_i \rangle = 0 \text{ when } i \neq j \right)$

$$\Longrightarrow c_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2}.$$

Prop Any set of mutually orthogonal vectors of size $\dim V$ is a basis.

If $\dim V = n$ and $v_1, \dots, v_n \ (\neq 0)$ are mutually orth., then they form a basis.

Pf Suffices to show $v_1, \dots, v_n$ are independent. Let

$$c_1 v_1 + \dots + c_n v_n = 0$$

Consider $\langle c_1 v_1 + \dots + c_n v_n, v_i \rangle = 0$

$$= \sum_{j=1}^{n} c_j \langle v_j, v_i \rangle = \sum \begin{cases} 0 & \text{if } i \neq j \\ c_i \|v_i\|^2 & \text{if } i = j \end{cases}$$

$$c_i \|v_i\|^2 = 0 \Rightarrow c_i = 0.$$

$\square$

Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$.

First of all, this is a mutually orthogonal set of vectors in $\mathbb{R}^4$.

$\Longrightarrow$ $\{v_1, \dots, v_4\}$ orthogonal basis.

__Ex__ Write $\begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix}$ as a linear combination of v_1, v_2, v_3, v_4.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix}$$

:(

Instead

$$c_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2}$$

$$c_1 = \frac{(4,-2,1,5)\cdot(1,1,1,1)}{4}$$

$$= \frac{8}{4} = \boxed{2}$$

$$c_2 = \frac{(4,-2,1,5)\cdot(1,1,-1,-1)}{4}$$

$$= \frac{-4}{4} = \boxed{-1}$$

$$c_3 = \frac{(4,-2,1,5)\cdot(1,-1,0,0)}{2} = \boxed{3}$$

$$c_4 = \frac{(4,-2,1,5)\cdot(0,0,1,-1)}{2} = \boxed{-2}$$

$$\begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

<u>Ex</u>

Consider $W = \mathrm{Span}(1, x, x^2)$

$\subseteq C^0[0,1] / \mathbb{R}$

$W =$ polynomials w/ deg 2 or less

$1, x, x^2$ is not an orthogonal basis!

$$\langle 1, x \rangle = \int_0^1 1 \cdot x \, dx = \frac{1}{2} \neq 0.$$

An orthogonal basis of this span is

$$\left\{ 1, \; x - \frac{1}{2}, \; x^2 - x + \frac{1}{6} \right\}$$

E.g. $\displaystyle\int_0^1 1 \cdot \left(x - \frac{1}{2}\right) dx = \frac{1}{2} - \frac{1}{2} = 0$

$= \langle 1, x - \frac{1}{2} \rangle$

<u>Ex</u> Write x^2+x+1 as a linear combination of $1,\ x-\frac{1}{2},\ x^2-x+\frac{1}{6}$.

$$c_1 = \frac{\langle x^2+x+1,\ 1\rangle}{\|1\|^2} = \frac{\int_0^1 (x^2+x+1)\cdot 1\,dx}{1}$$

$$= \frac{11}{6}$$

$$c_2 = \frac{\langle x^2+x+1,\ x-\frac{1}{2}\rangle}{\|x-\frac{1}{2}\|^2}$$

$$= \frac{\int_0^1 (x^2+x+1)(x-\frac{1}{2})\,dx}{\int_0^1 (x-\frac{1}{2})^2\,dx} = \frac{\frac{1}{6}}{\frac{1}{12}}$$

$$= 2$$

$$c_3 = \frac{\langle x^2+x+1,\ x^2-x+\frac{1}{6}\rangle}{\|x^2-x+\frac{1}{6}\|^2} = \frac{\frac{1}{180}}{\frac{1}{180}} = 1$$

$$x^2+x+1 = c_1(1) + c_2\left(x-\tfrac{1}{2}\right) + c_3\left(x^2-x+\tfrac{1}{6}\right)$$

$$= \frac{11}{6} + 2\left(x-\tfrac{1}{2}\right) + 1\left(x^2-x+\tfrac{1}{6}\right)$$

§ 4.2 Tomorrow ...

§ 4.3 Orthogonal Matrices

<u>Def</u> Let V be the vector space $\mathbb{R}^n$ w/ the dot product.

We say a matrix Q is <u>orthogonal</u> if $Q^TQ = QQ^T = I$, i.e. $Q^{-1} = Q^T$.

<u>Ex</u> $I \in M_{n\times n}(\mathbb{R})$ <u>Ex</u> $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is orthogonal

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$Q^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$Q^T Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $Q^T = Q^{-1}$, Q is orthogonal

Prop A matrix Q is orthogonal iff the columns form an orthonormal basis of $\mathbb{R}^n$ w/ dot product.

Pf If $Q = (q_1 \dots q_n)$ q_i = i^{th} column.

Suppose $Q^T Q = I$.

then
$$\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} (q_1 \dots q_n) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$(Q^T Q)_{ij} = q_i \cdot q_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Longrightarrow$ $q_i \cdot q_j = 0$ iff $i \neq j$

$\|q_i\|^2 = 1$ $\Rightarrow$ $q_1 \dots q_n$ orthonormal basis.

$(Q^T = Q^{-1})$

let $q_1 \dots q_n$ be an orthonormal basis.

let $Q = (q_1 \dots q_n)$.

$$Q^T Q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} (q_1 \dots q_n)$$

$$(Q^T Q)_{ij} = q_i \cdot q_j$$

since this is an orthon. basis

$$= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\Longrightarrow \quad Q^T Q = I. \quad \square$$

Prop If Q is orthogonal, then
$\det(Q) = \pm 1$.

Pf

$1 = \det(I)$

$= \det(Q^T Q)$

$= \det(Q^T)\det(Q)$

$= \det(Q)\det(Q)$ (from 1.9)

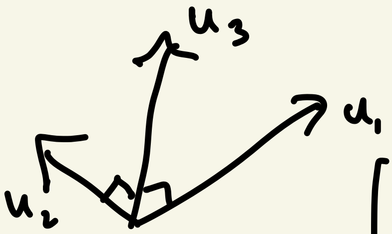
$= \det(Q)^2$

$\det(A^T) = \det(A)$

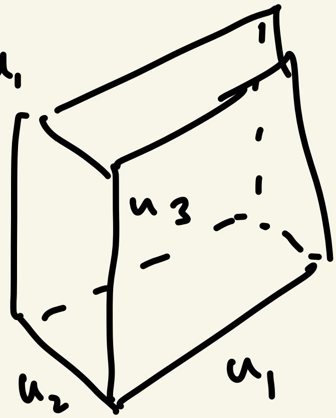
Take sq. rts.

$\det(Q) = \pm 1$.

Let $u_1 \, u_2 \, u_3$ be a orthonormal basis of $\mathbb{R}^3$. u_i is a unit vector and at 90° angles



$\Longrightarrow$



P

$$\mathrm{Vol}(P) = \left|\det\left(\left(u_1 \, u_2 \, u_3\right)\right)\right|$$

$$= |\pm 1| = 1$$

So P is actually some kind of cube.

Prop If P, Q are orthogonal, then so is PQ.

Pf It suffices to show that
$$(PQ)^T(PQ) = I.$$

(1.6)

$$\underbrace{(PQ)^T}(PQ)$$

$$= \underbrace{Q^T P^T}\cancel{P^T P}\, Q$$ P is orthog.

$$= \cancel{Q^T Q}$$ Q is orth.

$$= I.$$

So PQ is orthogonal.

Orthogonal matrices are a $\boxed{\text{Sub-object}}$ of invertible matrices.

object = group

like a subspace but w/ matrix multiplication

Orthogonal matrices preserve geometry.

$v, w \in \mathbb{R}^n$.

- let $d = \|v - w\|$
  $d' = \| Qv - Qw \|$

then $d = d'$.

Q preserves distances.

- let θ be angle between v, w
  θ' be the angle between Qv, Qw.

$\theta = \theta'$.

Q preserves angles.

Pf Need lemma.

Lemma Let Q be orthogonal.
Then $Qu \cdot Qv = u \cdot v$
$\forall\, u, v \in \mathbb{R}^n$.

Q preserves inner products.
(dot product here)

Pf lemma

$$Qu \cdot Qv = (Qu)^T (Qv)$$
$$= (u^T Q^T)(Qv)$$
$$= u^T (\cancel{Q^T Q})\, v \quad Q \text{ orth.}$$
$$= u^T v = u \cdot v$$

Pf main result

- Q preserves distances.

$$\|Qu - Qv\|^2$$

$$= \langle Qu - Qv, Qu - Qv \rangle$$ dot product

$$= \langle Q(u-v), Q(u-v) \rangle$$

$$= \langle u-v, u-v \rangle$$ by Lemma

$$= \|u-v\|^2.$$

- $$\theta' = \cos^{-1}\left(\frac{\langle Qu, Qv \rangle}{\|Qu\| \cdot \|Qv\|}\right)$$

$$= \cos^{-1}\left(\frac{Qu \cdot Qv}{\sqrt{(Qu \cdot Qu)(Qv \cdot Qv)}}\right)$$

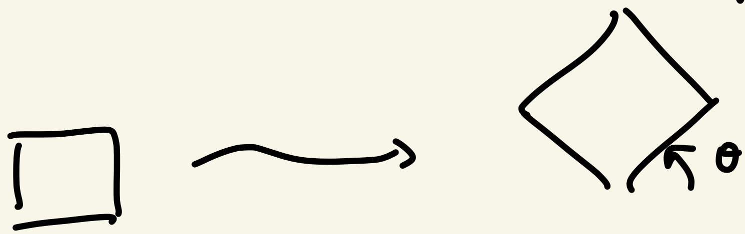
$$= \cos^{-1}\left(\frac{u \cdot v}{\|u\| \cdot \|v\|}\right) = \theta$$

Q is what's called an <u>isometry</u>, preserves angles and distances.

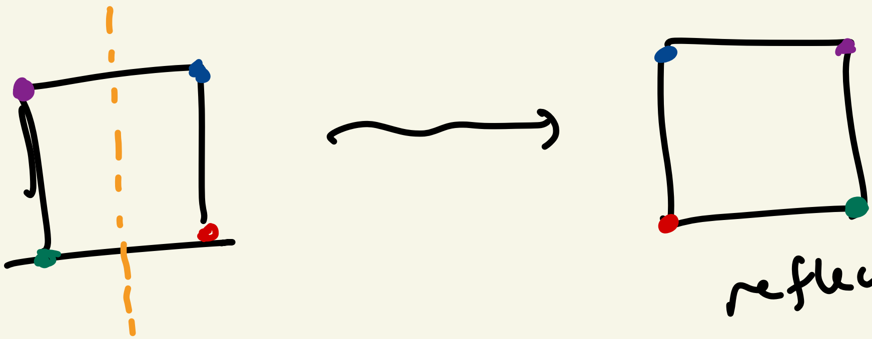
Isometry $\simeq$ rotation matrices

+ translations

reflections, rotations, translations



rotation



reflection

In $\mathbb{R}^2$ ...

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Q^T Q = I.$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a^2 + c^2 = 1 \quad *$$
$$b^2 + d^2 = 1$$
$$ab + cd = 0$$

(a, c) (b, d) are on the unit circle.

Any pt on the unit circle has the form $(\cos\theta, \sin\theta)$.

Let $a = \cos\theta$ $\quad b = \cos\varphi$

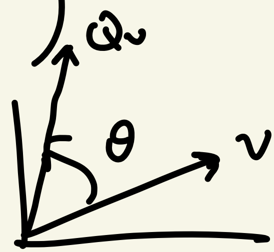
$c = \sin\theta$ $\quad d = \sin\varphi$

$$ab + cd$$
$$= \cos\theta\cos\varphi + \sin\theta\sin\varphi = 0$$
$$= \cos(\theta - \varphi) = 0$$
$$\theta - \varphi = \pm\frac{\pi}{2}$$
$$\varphi = \theta \pm \frac{\pi}{2}.$$

If $+$, $b = \cos\varphi = \cos\left(\theta + \frac{\pi}{2}\right)$
$$= -\sin\theta$$

$$d = \sin\varphi = \sin\left(\theta + \frac{\pi}{2}\right)$$
$$= \cos(\theta)$$

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ rotates by θ

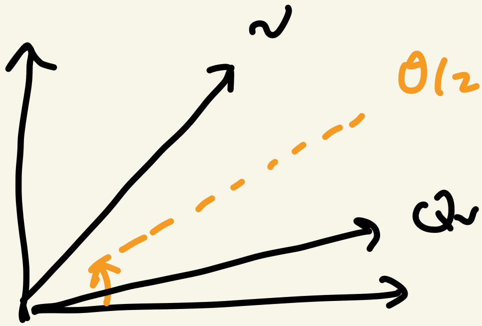


If $-$, $\varphi = \theta - \frac{\pi}{2}$

$$b = \cos\varphi = \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$
$$d = \sin\varphi = \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$$

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

reflects across the angle $\theta/2$.



Reflections in $\mathbb{R}^n$.

let H be a hyper-plane in $\mathbb{R}^n$

Hyper plane is a <u>subspace</u> that

solves

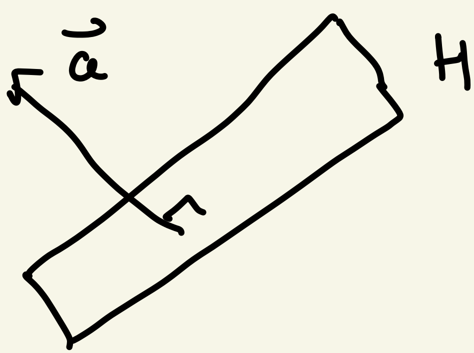
$$a_1x_1 + \dots + a_nx_n = 0.$$

$n-1$ dimensional subspace.

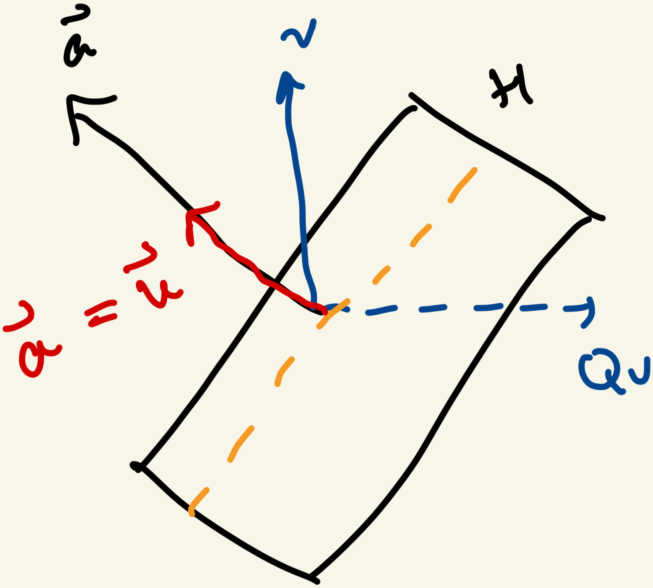
$$H \perp (a_1, \dots, a_n)$$

Since you can rewrite this to be

$$(a_1, \dots, a_n) \cdot (x_1, \dots, x_n) = 0.$$



Let's say we want to reflect across $\mathcal{H}$.



Let $Q = I - 2aa^T$.

a is a $n \times 1$ matrix

a^T is a $1 \times n$ matrix

a^Ta is 1×1

aa^T is $n \times n$.

$Q = I - 2aa^T$ is $n \times n$.

Prop $Q = I - 2aa^T$ is orthogonal

symmetric and it reflects

across H. $\vec{a}$ is a unit

vector $a^Ta = 1$

Pf

$Q^T = Q$

$\Rightarrow$ symmetric

Q^TQ

$= (I - 2aa^T)^T(I - 2aa^T)$

$= (I^T - (2aa^T)^T)(I - 2aa^T)$

$= (I - 2a^{T^T}a^T)(I - 2aa^T)$

$= (I - 2aa^T)(I - 2aa^T)$

$= I \cdot I - I(2aa^T) - 2aa^TI$

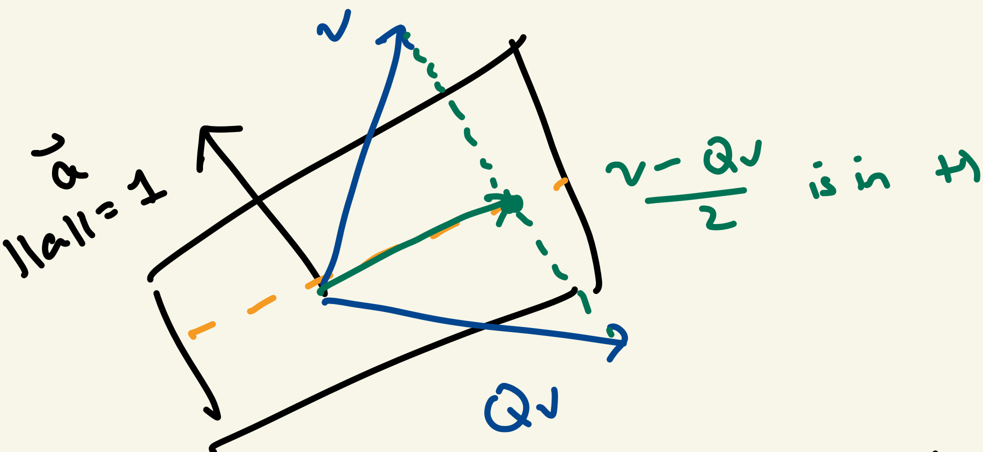
$+ 4aa^Taa^T$

$= I - 4aa^T + 4aa^Taa^T$

$$= I - 4aa^T - 4a\underbrace{\cancel{(a^Ta)}}_{1}a^T$$

$$= I - 4aa^T + 4aa^T = I$$

So $Q = I - 2aa^T$ is orthogonal, and symmetric.



To prove Q is a reflection, I have to specify what I mean by a reflection.

More general statement:

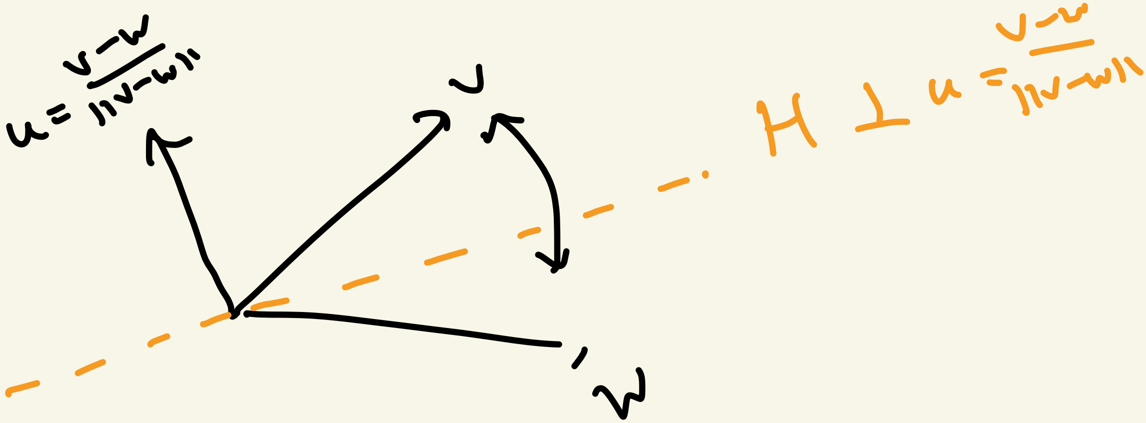
<u>Lemma</u> 4.28 Let $v, w \in \mathbb{R}^n$

w/ $\boxed{\|v\| = \|w\|}$

Let $a = \dfrac{(v-w)}{\|v-w\|}$.

Let $Q = I - 2aa^T$.

Then $Hv = w$ and $Hw = v$.



$Qv = w$, $Qw = v$.

Quick Pf

$$\boxed{Qv} = \left(I - 2\left(\frac{v-w}{\|v-w\|}\right)\left(\frac{v-w}{\|v-w\|}\right)^T\right)v$$

$$= \left(I - 2\frac{1}{\|v-w\|^2}(v-w)(v-w)^T\right)v$$

$$= \left(Iv - 2\frac{1}{\|v-w\|^2}(v-w)\underbrace{(v-w)^Tv}\right)$$

$$= \left(Iv - \frac{2}{\|v-w\|^2}\underline{(v-w)}(v^Tv - w^Tv)\right)$$

$$= \left(Iv - \frac{2(\|v\|^2 - v\cdot w)}{\underline{\|v\|^2} - 2v\cdot w + \underline{\|w\|^2}}(v-w)\right.$$

$$= \left(v - \frac{2\|v\|^2 - \cancel{2v\cdot w}}{2\|v\|^2 - 2v\cdot w}(v-w)\right)$$

$$= v - (v-w) = \boxed{w}$$

$\square$

In conclusion,

$$Q = I - 2aa^T \quad \|a\| = 1$$

reflects v across H perp. to a.

Q is orthogonal and symmetric.

Orthonormal basis w/ dot product

$\rightsquigarrow$ Q is orthogonal matrix.

$$Q^T Q = I.$$

Orthonormal basis in $\mathbb{R}^n$ w/ $\langle -,- \rangle$

$\rightsquigarrow$ $Q = ??$

Thm/Def: Let $\{q_1 \dots q_n\}$ be an orthon. basis w/r/t an $\langle -,- \rangle$ on $\mathbb{R}^n$.

Let $Q = (q_1 \dots q_n)$.

Then $\langle Qu, Qv \rangle = \langle u, v \rangle$.

$\left\{ \begin{array}{l} \underline{\text{Def}} \quad \text{All matrices } Q \text{ st} \\ \qquad \langle Qu, Qv \rangle = \langle u, v \rangle \ \forall u, v \\ \text{to be } \underline{\text{orthogonal}} \text{ w/r/t} \\ \qquad \langle -,- \rangle. \end{array} \right.$

$Q^T Q = I$ doesn't generalize to arbitrary inner products.

But $Qu \cdot Qv = u \cdot v$ does generalize.

Thm Let Q be a matrix.

Then Q is orthogonal ($Q^T Q = I$)

iff $Qu \cdot Qv = u \cdot v \quad \forall u, v \in \mathbb{R}^n$.

We already proved that

if $Q^T Q = I$

$\Longrightarrow$ Q preserves dot products.

Pf

Assume $\forall u, v \in \mathbb{R}^n$

that $Qu \cdot Qv = u \cdot v$.

Pick $u = e_i \quad v = e_j$.

Then $Qu \cdot Qv = Qe_i \cdot Qe_j = e_i \cdot e_j$

$= q_i \cdot q_j = e_i \cdot e_j$

$q_i \cdot q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \|q_i\| = 1$

Therefore the columns of Q
form an orthonormal basis
$\implies$ Q is orthogonal.

<u>Def</u> of orth. $Q^T Q = I$
$\iff$ columns of Q form
orth basis.

<u>Thm</u> $\det(Q) = \pm 1$

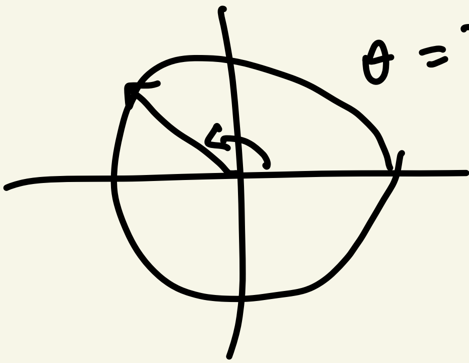
<u>Thm</u> PQ is also orth.

2×2 orth. matrices

$Q = I - 2aa^T$ reflection matrix
in $\mathbb{R}^n$

$\|a\| = 1.$

$$e^{2\pi i a} = e^{(2\pi a) i}$$



$$e^{2\pi a i} \overset{\times}{=} \left(e^{2\pi i}\right)^{a} = 1^{a} \overset{\times}{=} 1$$

$a = \frac{1}{2}$

$$e^{2\pi \frac{1}{2} i} = e^{\pi i} = -1$$

$$\left(e^{2\pi i}\right)^{1/2} = 1^{1/2} = \pm 1$$

z^a is not well defined for $a \notin \mathbb{Z}$.

$$e^{2x} = a \cdot 1 + be^{x}$$

$$2e^{2x} = be^{x}$$

$$1, e^{x}, e^{2x}$$

$$K = (k_1 \; k_2 \; k_3)$$

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

$\det(2) = 2 > 0$

$\det\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} = 9 > 0 \Rightarrow K$ pos def

K is positive def.

$$\boxed{x^T K^2 x > 0}$$ WTS

$K^T = K$

$$x^T K^2 x = x^T K K x$$

$$= (x^T K^T) K x$$

$$= (Kx)^T Kx = Kx \cdot Kx$$

$$= \|Kx\|^2 > 0$$

as long as $x \neq 0$.
K is nonsingular

$$x \cdot y = x^T y$$

$$z = x + iy \qquad \rightsquigarrow \qquad \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$$

$$w = u + iv$$

$$\mathbb{C}/\mathbb{R}$$

$$\operatorname{Re}(z\bar{w}) = 0 \iff (x,y) \perp (u,v)$$

$$\Updownarrow$$

$$xu + yv = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{bmatrix} a & c \\ c & d \end{bmatrix}$$

$\underline{a > 0}$ and $\boxed{\underline{ad - c^2 > 0}}$ $+$

$$ax^2 + 2cxy + dy^2 > 0$$

$$x = \frac{-2cy \pm \sqrt{4c^2y^2 - 4ady^2}}{2a}$$

$$4c^2y^2 - 4ady^2 < 0$$

$$4y^2(c^2 - ad) < 0$$

$$c^2 - ad < 0$$

$$ad - c^2 > 0$$

$$\Rightarrow \quad ax^2 + 2cxy + d^2y^2 > 0$$

$$\begin{pmatrix} -2+i \\ i \end{pmatrix}, \begin{pmatrix} 4-3i \\ 1 \end{pmatrix}, \begin{pmatrix} 2i \\ 1-5i \end{pmatrix}$$

dependent

(f) $\begin{pmatrix} 1 & 1+2i & 1-i \\ 3i & -3 & -i \\ 2-i & 0 & 1 \end{pmatrix}$

$\det \neq 0 \Rightarrow$ independent

$r_2' = -3i\, r_1 + r_2$

$(-3i)(1+2i)$

$-3i + 6 + -3$

-3

$$\begin{pmatrix} 1 & 1+2i & 1-i \\ 0 & 3-3i & 3-4i \\ 2-i & 0 & 1 \end{pmatrix}$$

$(-3i)(1-i)$

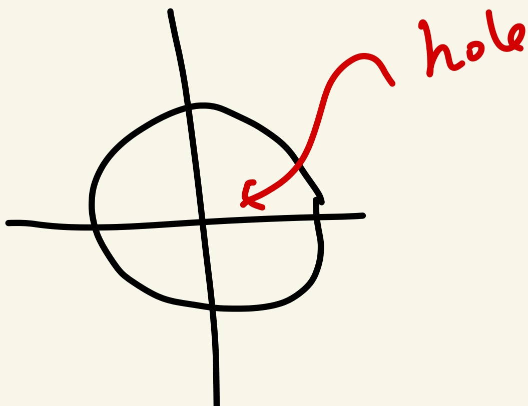
$-3i + 3 - i$

$3 - 4i$

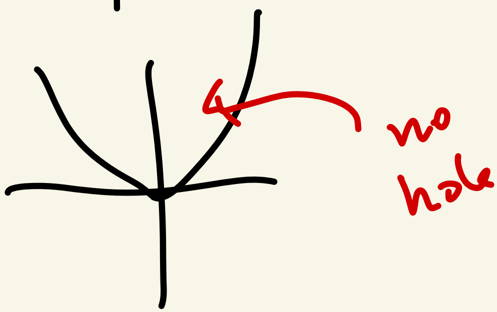
$$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \to \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$$

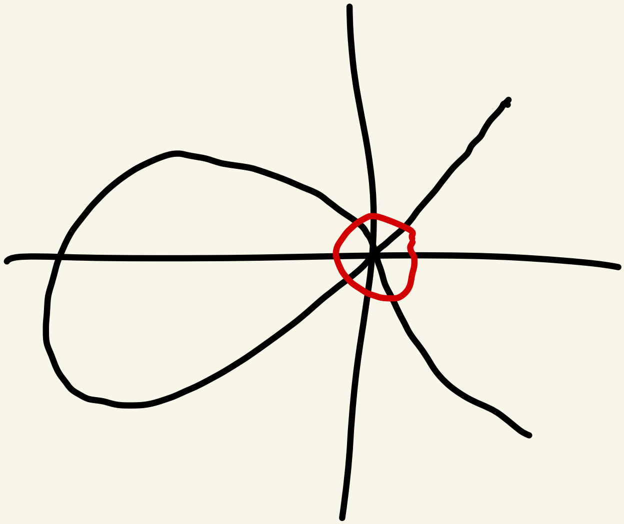
$$\to \begin{pmatrix} -1 & i \\ 0 & 2i \end{pmatrix}$$

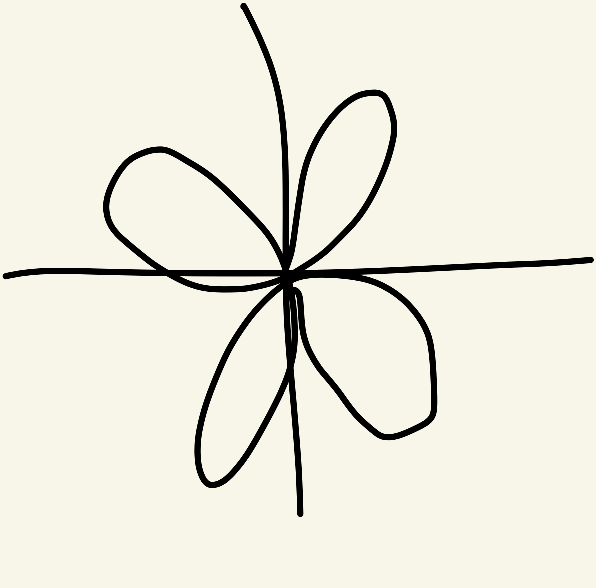
$x^2 + y^2 - 1 = 0$ $\qquad$ $x^p + y^p = 1$

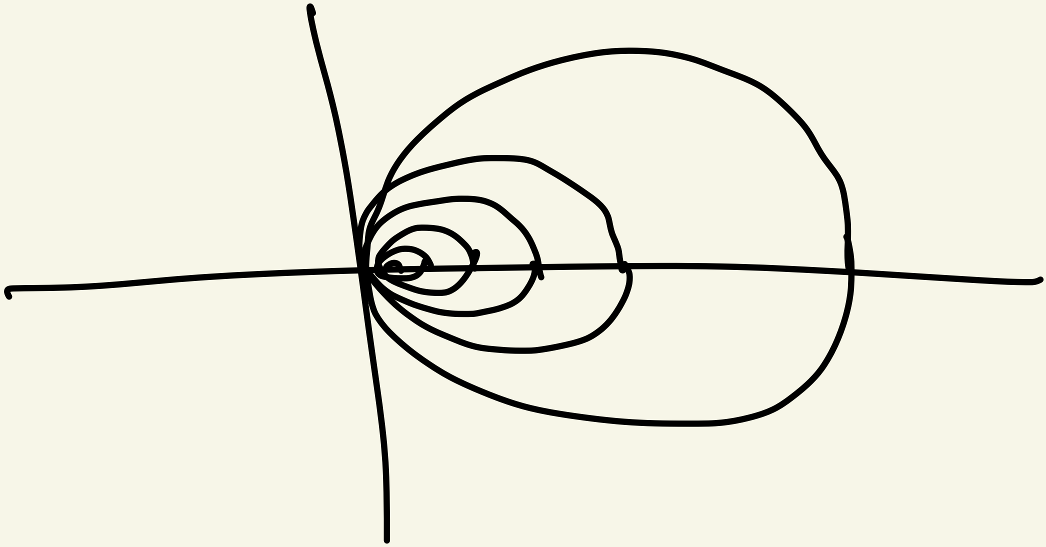


$y - x^2 = 0$









Hawaiian Earring

(a) $X^T K^{-1} x \qquad x = Ky$

$= (Ky)^T \cancel{K^{-1} (Ky)}$

$= (Ky)^T y$

1.6

$(AB)^T$

$= B^T A^T$

$= y^T K^T y$

$= y^T K y$

(b) $x^T K^{-1} x > 0$?

$\downarrow$

$y^T K y > 0 \Rightarrow x^T K^{-1} x > 0$

3.4.30

Since $K^T = K$

$$x^T K^2 x = x^T K^T K x$$

$$= \dots$$