§ 1.2. Matrices

$M_{n \times m}(\mathbb{R})$

$=$ set of $n \times m$ matrices w/ entries in $\mathbb{R}$

n rows

m columns

$M_{n \times m}(\mathbb{C}) = \dots$ in $\mathbb{C}$

$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$

$$\begin{pmatrix} i & 5 \\ -3+i & i-1 \end{pmatrix}$$

Linear systems turn out to be matrix equations of the form $A\vec{x} = \vec{b}$.

Matrix Multiplication

Given $A \in M_{\boxed{n} \times \cancel{m}}(\mathbb{R})$

$B \in M_{\cancel{m} \times \boxed{p}}(\mathbb{R})$

we can define $AB \in M_{n \times p}(\mathbb{R})$

If a_{ij} is the ij entry of A

b_{ij} is the ij entry of B

then $\boxed{(AB)_{ij}} = \sum_{k=1}^{m} (a_{ik})(b_{kj})$

$= \sum_{k=1}^{m} (A)_{ik} (B)_{kj}$

Ex let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \quad \boxed{2} \times \cancel{3}$$

$$B = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \quad \cancel{3} \times \boxed{1}$$

$AB \ (2 \times 1)$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \cdot 5 + 1 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 5 + (-1) \cdot 1 + 2 \cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 11 \\ -1 \end{pmatrix}$$

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = \begin{pmatrix} \boxed{b_1} \\ \vdots \\ b_m \end{pmatrix}$$

$$\langle a_{i1}, x_i \rangle = \sum a_{i1} x_i$$

$$= a_{11}x_1 + \dots + a_{in}x_n$$

$$= b_1$$

If we let $A = \begin{pmatrix} a_{11} & \cdots & \\ \vdots & \ddots & \\ \vdots & \cdots & a_{mn} \end{pmatrix}$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Liner System

$$\Rightarrow \quad Ax = b.$$

Looking Ahead, if

$$5x = 10$$

$$x = \frac{10}{5} = 2.$$

Is there a way to
"divide by A"?

Most of the time!

Matrices also add.

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 \\ 5 & 2 \end{pmatrix}$$

Scalar Multiplication

if $c \in \mathbb{R}, \mathbb{C}$ cA is defined as

$$(cA)_{ij} = ca_{ij}$$

$$2\begin{pmatrix}5 & 1\\ -3 & 1\end{pmatrix} = \begin{pmatrix}10 & 2\\ -6 & 2\end{pmatrix}$$

Note!

$$A - B = A + (-1\cdot B)$$

A column vector is an $n\times 1$ matrix.

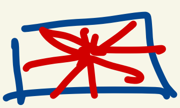
$$\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}$$

or

$$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}.$$

A row vector is a $1\times m$ matrix

$$(a_1 \cdots a_m)$$



$$(x_1 \cdots x_m)A = (b_1 \cdots b_m)$$

- $A_{n \times n}$ is called a square matrix.

- A matrix U is called upper triangular if $a_{ij} = 0$ if $i > j$.

$$\begin{pmatrix} a_{11} & & & a_{ij} \\ & a_{22} & & j>i \\ & & a_{33} & \\ a_{ij} = 0 & & & \ddots \\ (i>j) & & & & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

- Lower triangular matrices

L is lower triangular if $l_{ij} = 0$ when $j > i$.

$$\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ * & & a_{nn} \end{pmatrix} = \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}$$

- A matrix is <u>diagonal</u> if

$a_{ij} = 0$ $i \neq j$.
$$\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

Note: If A is both upper and lower triangular, it is diagonal.

Fact: Matrix Multiplication is not commutative.

$\boxed{AB \neq BA}$ in general!

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$$

$Ax = b$

Alternate formula

$B = \left(\vec{b}_1 \cdot \ldots \vec{b}_n \right)$ $n \times n$

(A is also $n \times n$)

$AB = \left(A\vec{b}_1 \ldots A\vec{b}_n \right)$

$(AB)_i$ = ith column of AB

by def = $A b_i$.

a_i is the ith row of A

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} B = \begin{pmatrix} a_1 B \\ \vdots \\ a_n B \end{pmatrix}$$

$A \cdot A = A^2$ if a_i is the i^{th} row

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} a_1^T & \cdots & a_n^T \end{pmatrix}$$

$$\left.\begin{array}{l} a_{i*} = i^{th} \text{ row} \\ \\ a_{*j} = j^{th} \text{ column} \end{array}\right\}$$

$AB \neq BA$ makes some sense since (a_i = i^{th} column)

$$\begin{pmatrix} Ab_1 & \cdots & Ab_n \end{pmatrix} \neq \begin{pmatrix} Ba_1 & \cdots & Ba_n \end{pmatrix}$$

- Identity matrix.

Define the $n \times n$ identity matrix I_n to be the diagonal matrix w/ $(I_n)_{ii} = 1$

$$\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Prop
If A is $m \times n$, then

$$AI_n = A$$

When A is square ($n \times n$)

$$A I_n = A \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 1 \end{pmatrix}$$

$$= \left(A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \; A\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots A\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} \quad \text{etc}$$

$$= AI_n = \begin{pmatrix} a_{11} & a_{21} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \end{pmatrix}$$

$$= A$$

(real number mult. , 1)

$$a \cdot 1 = a$$

(matrix mult. $n \times n$, I_n)

$$A \cdot I_n = A$$

$$I_n \cdot A = A$$

Table of Properties

- $A + B = B + A$
- $(A+B)+C = A+(B+C)$
- $c(A+B) = cA + cB, \quad c \in \mathbb{R}$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$
- $(AB)C = A(BC)$
- $(cA)B = c(AB) = A(cB)$
- $AI_n = I_nA = A \quad (n \times n)$
- $A + 0 = 0 + A = A$
- $0A = A0 = 0$

Problem Set

(1) Let U, V be upper Δ.

then $u_{ij} = 0$ if $i > j$

$v_{ij} = 0$ if $i > j$.

$$(UV)_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj}$$

If $i > j$, then

either $k < i$ or

$k > j$.

$$\{\overbrace{1 \ldots j \ldots i}\ldots n\}$$

$$\{1 \ldots \underbrace{j \ldots i \ldots n}\}$$

If $k < i$ then $u_{ik} = 0$

If $k > j$ then $v_{kj} = 0$

Since either $u_{ik} = 0$ or $v_{kj} = 0$

$\forall k$.

then $\sum_{k=1}^{n} u_{ik} v_{kj} = 0$

$\Rightarrow (UV)_{ij} = 0$

if $i > j$

UV is upper Δ

② $AX = I_2$

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

we can now reduce this!

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-2r_1+r_2} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

$$-r_2+r_1 \quad \left(\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

$$X = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

$(\boxed{2}\times\cancel{2})(\cancel{n}\times\boxed{m})$

$= (2\times\boxed{2})$

$n=2$

$m=2$

yes $\underline{2\times 2}$

$$[[A,B],C] + [[C,A],B] + [[B,C],A]$$

$$= A\cancel{BC} - \cancel{BA}C - C\cancel{AB} + C\cancel{BA}$$

$$+ C\cancel{AB} - A\cancel{CB} - B\cancel{CA} + \cancel{BA}C$$

$$+ \cancel{BC}A - \cancel{CB}A - A\cancel{BC}$$

$$+ A\cancel{CB} = 0$$

$[\,,\,]$

Jacobi ID $\rightsquigarrow$ Lie algebra

$\uparrow$ $\dfrac{\partial}{\partial x}$

particle physics $\xleftarrow{\text{Linear Alg}}$ Lie Groups

§1.3 Gaussian Elimination
— Regular Case

System of Linear Eq'ns $\rightsquigarrow$ Augmented Matrix

At the end of row reduction example yesterday.

$$\left(\begin{matrix} * & & * \\ 0 & * & \\ 0 & 0 & * \end{matrix} \middle| \begin{matrix} \\ \vdots \\ \\ \end{matrix}\right)$$

$$= \left(U \mid c \right)$$

where U is upper Δ.

$$U\vec{x} = \vec{c}.$$

$$u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = c_{n-1}$$

$$u_{nn}x_n = c_n$$

$$x_n = \frac{c_n}{\boxed{u_{nn}}} \neq 0$$

Back substitution

$$x_{n-1} = \text{----}$$

Systems $Ux = c$ are solvable by back substitution!

$$Ax = b \rightsquigarrow Ux = c.$$

best case scenario.

$$u_{ii} \neq 0.$$

A square matrix A is called **regular** if

A can be row reduced to an upper Δ matrix U such that

$U_{ii} \neq 0$ and no row swapping when reducing.

(U is sometimes called reduced form, or reduced echelon form)

$$\begin{pmatrix} * \end{pmatrix} \longrightarrow \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

When you row reduce

$$A \longrightarrow U$$

in the regular case

all row operations should be of the form

$$r_i' = c r_i$$

or $r_j' = c r_i + r_j$

$j > i$.

$$\begin{pmatrix} * & & \\ 0 & * & \\ 0 & 0 & * \end{pmatrix}$$

Side note:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$\begin{pmatrix} c\,\boxed{a_1} \\ \vdots \\ -ca_1 + a_n \end{pmatrix}$$

$$a_1' = c\,a_1 + \underline{a_n'}$$

$$a_n' = a_n - a_1$$

$$a_1' = a_1 + a_n'$$

$$= a_1 + (a_n - a_1)$$

$$= a_n$$

Elementary matrices

Let's say we have a row operation

$$r_j' = \underline{c}\,r_i + r_j. \qquad R$$

Apply R to I_n

$$\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}$$

$$\xrightarrow{R} \begin{array}{c} \\ \\ \\ \\ j \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & i & & & & \end{array} \\ \begin{pmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ 0 & & \underline{c} & & 1 & & \\ & & & & & \ddots & \\ & & 0 & & & & 1 \end{pmatrix} \end{array} = E_R$$

$$a'_{ji} = c$$

E_R is the elementary matrix

corresponding to R

$(= r_j' = cr_i + r_j)$

For every row operation ρ

E_ρ is the matrix

obtained by apply ρ

to I_n.

$r_j' = cr_i + r_j$ swap(i, j)

$$\begin{array}{cc} & \begin{matrix} & i & & \end{matrix} \\ \begin{matrix} \\ \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & c & 1 & \\ & & & 1 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \begin{matrix} & & i & & j & \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & 1 & \\ & & & 1 & & \\ & & 1 & & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$r_i' = c r_i$

$$\begin{array}{cc} & \begin{matrix} & & i & & \end{matrix} \\ i & \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \end{array}$$

Proposition: Given a row operation ρ, $A \xrightarrow[\rho]{} A'$, and E_ρ is the elementary matrix corresponding to ρ (rho).

Then $A' = E_\rho A$.

Doing the row operation is the same as multiplying by the elementary matrix!

$$A \xrightarrow{\ell_1} A' \xrightarrow{\ell_2} \cdots\cdots \xrightarrow{\ell_m} U$$

$$A' = E_{\ell_1} A$$

$$A'' = E_{\ell_2} E_{\ell_1} A$$

$$\vdots$$

$$U = E_{\ell_m} E_{\ell_{m-1}} \cdots\cdots E_{\ell_1} A$$

Pf Check each case.

Ex

$$\begin{pmatrix} -1 & -1 & 3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \xrightarrow[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{r_1' = -1r_1} \begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]{-2r_1 + r_2} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 3 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}]{-3r_1 + r_3} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & -3 & 7 \end{pmatrix}$$

$$\xrightarrow[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}]{-r_2 + r_3} \begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$$

Claim:

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}} \underline{\begin{pmatrix} -1 & -1 & 3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}}$$

If you multiply a lower Δ matrix by another, you get a 3rd lower Δ matrix.

Since $j>i$ in $cr_i + r_j$ then E's are lower Δ.

If A is regular,

$$U = \underbrace{E_m \cdots\cdots E_1} A$$

then $E_m \cdots\cdots E_1 = L_1$

is lower Δ'.

$$U = L_1 A.$$

<u>Ex</u>

$$\begin{pmatrix} 1 & 1 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & -1 \end{pmatrix} = L_1 \begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix}$$

But we can do better!

$$U = L_1 A$$

Goal! We want to decompose a regular matrix into $A = LU$.

Recall: $n = p_1^{\alpha_1} \cdots p^{\alpha_m}$

(prime factor decomposition)

If we want to understand how A row reduces, we want to decompose it into $A = LU$.

HW 1 due at
the end of today!
11:59.

HW2 is up on canvas