§ 4.2 Gram-Schmidt Process

How do you actually compute orthogonal bases and orthonormal bases?

Most bases are neither orthogonal or orthonormal!

But there is a process, called the Gram-Schmidt process, which takes any basis

$$\{w_1, \ldots, w_n\} \longrightarrow \{v_1, \ldots, v_n\}$$

and gives you an orthogonal basis.

Method:

Given $\{w_1, \dots, w_n\}$

construct $v_1, \dots, v_n$ recursively

1. let $v_1 = w_1$.

2. let $v_2 = w_2 - cv_1$.

Hope that we can find a c

such that $\langle v_2, v_1 \rangle = 0$.

$$\langle v_2, v_1 \rangle = \langle w_2 - cv_1, v_1 \rangle = 0$$

Solve for c.

$$\langle w_2, v_1 \rangle - c \langle v_1, v_1 \rangle = 0$$

$$c = \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2}$$

Therefore

$$\text{Let} \quad v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

This is the 2^{nd} basis vector in our orthogonal basis.

known

③ $$\text{let} \quad v_3 = \overset{\frown}{w_3} - \underline{c_1} \overset{\frown}{v_1} - \underline{c_2} \overset{\frown}{v_2}$$

Solve for c_1, c_2 to find v_3.

In theory

$$\langle v_3, v_1 \rangle = 0 \rightsquigarrow c_1$$

$$\langle v_3, v_2 \rangle = 0 \rightsquigarrow c_2$$

$$\langle v_3, v_1 \rangle = \langle w_3 - c_1 v_1 - c_2 v_2, v_1 \rangle = 0$$

$$= \langle w_3, v_1 \rangle - c_1 \|v_1\|^2 - c_2 \underbrace{\langle v_2, v_1 \rangle}_{=0}$$

$$= 0$$

$$\Rightarrow \langle w_3, v_1 \rangle = c_1 \|v_1\|^2$$

$$c_1 = \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2}$$

$$\langle v_3, v_2 \rangle = 0$$

$$\langle w_3 - c_1 v_1 - c_2 v_2, v_2 \rangle = 0$$

$$\langle w_3, v_2 \rangle - c_1 \cancel{\langle v_1, v_2 \rangle} - c_2 \langle v_2, v_2 \rangle = 0$$

$$c_2 = \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2}$$

In conclusion

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

v_3 is the 3rd orthogonal basis vector.

Gram-Schmidt says do this until you're done.

In general, Given $v_1, \dots, v_{i-1}$

at i^{th} step

$$v_i = w_i - \frac{\langle w_i, v_1 \rangle}{\|v_1\|^2} v_1 - \dots - \frac{\langle w_i, v_{i-1} \rangle}{\| v_{i-1} \|^2} v_i$$

$$v_i = w_i - \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\| v_j \|^2} v_j$$

$\rightsquigarrow$ $\{v_1, v_2, \dots, v_n\}$ is orthogonal by construction.

$\rightsquigarrow$ $\left\{ \frac{1}{\|v_1\|} v_1, \dots, \frac{1}{\|v_n\|} v_n \right\}$ is orthonormal.

Ex $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $w_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$\mathbb{R}^3$ w/ dot product

Not orthogonal. $\langle w_1, w_2 \rangle$
$= w_1 \cdot w_2 = 1 \neq 0.$

① $v_1 = w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

② $v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{(1 \;\; 1 \;\; -1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\|(1,1,1)\|^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = v_2$$

Note $v_1 \cdot v_2 = 0$

③ $v_3 = w_3 - \frac{w_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{w_3 \cdot v_2}{\|v_2\|^2} v_2$

$$= \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \frac{(0\ {-1}\ 2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3}\frac{(0\ {-1}\ 2)\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}}{\left(\frac{24}{9}\right)} v_2$$

$$= \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{9} \cdot \frac{9}{24} \cdot (-10) \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{5}{12}\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} \quad v_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

orthog.

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{24}}\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

orthonormal

$$Q = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \end{pmatrix}$$

is an orthogonal matrix!

$$v_i = w_i - \sum_{j=1}^{i-1} c_j v_j$$

c_j known

$$c_j = \frac{\langle w_i, v_j \rangle}{\|v_j\|^2}$$

$$w_i = v_i + \sum_{j=1}^{i-1} c_j v_j$$

$$w_1 = v_1$$

$$w_2 = v_2 + c_{12} v_1$$

$$w_3 = v_3 + c_{13} v_1 + c_{23} v_2$$

$$\vdots$$

We used back substitution to solve for $v_1 \dots v_n$.

Alternate Gram-Schmidt

$\{w_1, \dots, w_n\} \rightsquigarrow \{u_1, \dots, u_n\}$

Assumption

1 2 3

$r_{12} = \langle w_2, u_1 \rangle$

$r_{22} = \sqrt{\|w_2\| - r_{12}^2}$

$u_2 = \dfrac{w_2 - r_{12}u_1}{r_{22}}$

$w_1 = r_{11}u_1$

$w_2 = r_{12}u_1 + r_{22}u_2$

$w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3$

$\vdots$

$w_n = r_{1n}u_1 + \dots + r_{nn}u_n$

Use back substitution solve for u_1 then u_2, then u_3, etc.

1) $w_1 = r_{11}u_1$. Since $\|u_1\| = 1$

$\Rightarrow r_{11} = \|w_1\|$.

$$u_1 = \frac{w_1}{\|w_1\|}$$

(2) $w_2 = \boxed{r_{12}} u_1 + \underline{r_{22}} u_2$

First solve for r_{12} and r_{22}.

$$\langle w_2, u_1 \rangle = r_{12} \langle u_1, u_1 \rangle^{1} + r_{22} \cancel{\langle u_2, u_1 \rangle}$$

$$\langle w_2, u_1 \rangle = r_{12}$$

Since $u_1, u_2 \ldots u_n$ is supposed to be orthonormal

$$\|w_2\|^2 = r_{12}^2 + r_{22}^2$$

$$\Rightarrow r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2}$$

$$u_2 = \frac{w_2 - r_{12} u_1}{r_{22}}$$

In general... if $i < j$ Trying to find u_j

$$r_{ij} = \langle w_j, u_i \rangle$$

$$r_{jj} = \sqrt{\|w_j\|^2 - r_{ij}^2 - \ldots - r_{j\,j-1}^2}$$

$$u_j = \frac{w_j - r_{1j} u_1 - \ldots r_{j-1,j} u_{j-1}}{r_{jj}}$$

Ex $\quad w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

① $w_1 = r_{11} u_1$

$$r_{11} = \|w_1\| = \sqrt{3}$$

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

② $w_2 = \underline{r_{12}} u_1 + \underline{r_{22}} u_2$ *

$$r_{12} = w_2 \cdot u_1 = (1 \;\; 1 \; -1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2} = \sqrt{3 - \frac{1}{3}}$$

$$= 2\sqrt{\frac{2}{3}}$$

$$u_2 = \frac{w_2 - r_{12} u_1}{r_{22}} = \frac{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)}{2\sqrt{\frac{2}{3}}}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

③ $w_3 = \underline{r_{13}} u_1 + \underline{r_{23}} u_2 + \underline{r_{33}} \underline{u_3}$

$w_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$$r_{13} = w_3 \cdot u_1 = (0\ -1\ 2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$$r_{23} = w_3 \cdot u_2 = (0\ -1\ 2)\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\left(\frac{1}{\sqrt{6}}\right)$$

$$= \frac{-5}{\sqrt{6}}$$

$$r_{33} = \sqrt{\|w_3\|^2 - r_{13}^2 - r_{23}^2}$$

$$= \sqrt{5 - \frac{1}{3} - \frac{25}{6}} = \frac{1}{\sqrt{2}}$$

$$u_3 = \frac{w_3 - r_{13} u_1 - r_{23} u_2}{r_{33}}$$

$$= \frac{\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) - \frac{-5}{\sqrt{6}} \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

So $u_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ $u_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$

$u_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{pmatrix}$

Let $A = (w_1 \; w_2 \; w_3)$

$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$w_1 = r_{11} u_1$

$w_2 = r_{12} u_1 + r_{22} u_2$

$w_3 = r_{13} u_1 + r_{23} u_2 + r_{33}$

$\longrightarrow \quad A = (u_1 \; u_2 \; u_3) \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{pmatrix}$

$A = QR$