Chapter 8

Def: We say λ is an eigenvalue of a square matrix A if $\exists$ a nonzero vector v, called the eigenvector, such that

$$Av = \lambda v.$$

So in the direction of v, multiplying by A just scales by λ.

Prop Let A be an $n \times n$ matrix.

Then λ, v are an eigenvalue / eigenvector

of A iff

$$\det(A - \lambda I) = 0$$

$*$ $\ker(A - \lambda I)$ is non trivial.

Pf: By def $Av = \lambda v$

$$\iff Av - \lambda v = 0$$

$$\iff Av - \lambda I v = 0$$

$$\iff (A - \lambda I)v = 0$$

since $v \neq 0$

$\ker(A - \lambda I)$ is nontrivial

A square matrix B is invertible iff

- $\ker(B) = 0$
- $\det(B) \neq 0$
- columns are independent.

$\Longleftrightarrow$ $\ker(A - \lambda I) \neq 0$

by theory of square matrices $\Longleftrightarrow$ $\det(A - \lambda I) = 0$ $\square$

So the eigenvalues of A are the solutions to $\det(A - \lambda I) = 0$.

then find the eigenvector v by solving for $\ker(A - \lambda I)$.

row reduction

Ex $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

The eigenvalues are solutions to
$\det(A - \lambda I) = 0.$

$$\det\left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad *$$

$$= \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)^2 - 1 \cdot 1 = (3-\lambda)^2 - 1 = 0$$

$$\to (3-\lambda)^2 - 1 = 0$$

$$9 - 6\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 4) = 0$$

$$\underline{\lambda = 2}, \ \underline{\lambda = 4}$$

$\lambda = 2$

$\ker(A - 2I) \quad (\lambda = 2)$

$$A - 2I = \begin{pmatrix} 3-2 & 1 \\ 1 & 3-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{array}{cc} & \text{free} \\ \end{array} \begin{pmatrix} \underline{1} & 1 \\ 0 & 0 \end{pmatrix}$$

y is free

$x + y = 0 \quad$ y is free

$x = -y$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} y$$

$\Rightarrow \quad v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a eigenvector corresponding to $\lambda = 2$.

$\lambda = 4$

$$A - 4I = \begin{pmatrix} 3-4 & 1 \\ 1 & 3-4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$x - y = 0 \implies x = y$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y$$

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a eigenvector corresponding to $\lambda = 4$

$\lambda = 2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad\quad$ $\lambda = 4, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_2 \quad \hat{2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad\quad$ $v_4 \quad \hat{4} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Ex $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 0-\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$= \underline{-}\lambda^3 + 4\lambda^2 - 5\lambda + \underline{2} = 0$$

How to find λ?

If λ is an integer, then -1 and 2 determine that $\lambda = \underline{\pm 1, \pm 2}$

$\Rightarrow$ use polynomial long division

$\det(A - 1I)$

$$= -(1)^3 + 4 - 5 + 2$$

$$= 0 \qquad \lambda = 1 \text{ is a root!}$$

$\lambda - 1$ divides $-\lambda^3 + 4\lambda^2 - 5\lambda + 2$

$$\lambda - 1 \overline{\smash{)}\ -\lambda^3 + 4\lambda - 5\lambda + 2}$$

$$= -(\lambda-1)(\lambda-2)$$

$\underline{\lambda = 1}$ $\quad$ $\underline{\lambda = 1}$ $\quad$ $\lambda = 2$

$$\det(A - \lambda I) = -(\lambda-1)^2(\lambda-2) = 0$$

Is there a distinct eigenvector for each $\lambda = 1$?

In this case yes, but not in general. (For tomorrow)

$\lambda = 1$

$$A - 1I = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x + y + z = 0$$

$$\ker(A - 1I) = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = \underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} y + \underline{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} z$$

$\lambda = 1$ $\qquad$ $\lambda = 1$

$$v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad v = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 2$

$$A - 2I = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{reduce}]{\text{row}} \begin{array}{c} \hphantom{1 \quad 0 \quad} \text{free} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{aligned} x + z &= 0 \\ y - z &= 0 \end{aligned}$$

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} z$$

$$\underline{v = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}, \quad \underline{\lambda = 2}$$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \underline{2} \, \underline{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

$$Av = \lambda v$$

Snags:

① In general

$\det(A - \lambda I)$ is an n-deg polynomial,

find λ by factoring

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$\lambda_1 \ldots \lambda_n$ are eigenvalues

λ_i might repeat, there may not be enough eigenvectors if the λ_i repeat.

Def: Let A be an $n \times n$ matrix. Then the $\det(A - \lambda I)$ is called the <u>characteristic polynomial</u>

$$P_A(\lambda) = \det(A - \lambda I).$$

- $\deg(P_A(\lambda)) = n.$
- $P_A(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_r)^{k_r}$

We call the number of times λ_i repeats the <u>algebraic multiplicity</u> of the eigenvalue.

$(k_i).$

Def: Let A be $n \times n$ w/ eigenvalue λ.

Define $V_\lambda = \ker(A - \lambda I) \neq 0$.

$$= 0 \cup \left\{ \begin{array}{l} \text{all possible choices} \\ \text{of eigenvector} \\ \text{of } \lambda \end{array} \right\}$$

Ex $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\lambda = 1 \rightsquigarrow$ alg mult $= 2$

$$V_1 = \ker(A - 1 I)$$

$$= \operatorname{span}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right) \quad \dim(V_1) = 2$$

Prop Algebraic mult of λ $\geq \dim(V_\lambda)$

(2) $\det(A - \lambda I)$ might have complex solutions even though A was a real matrix.

<u>Ex</u> $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ = rotation matrix for $90° = \frac{\pi}{2}$

$\det(A - \lambda I) =$

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)^2 - (-1)(1)$$
$$= \lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\Rightarrow \lambda = \underline{\pm i}$$

$V_i = \ker(A - iI)$

$= \ker \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$ $x = iy$

$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} y$ $V_i = \underline{\text{span}\begin{pmatrix} i \\ 1 \end{pmatrix}}$

$$V_{-i} = \ker\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} = \operatorname{span}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

Recall: If $\lambda = \alpha + i\beta$ is a solution to real polynomial, then $\bar{\lambda} = \alpha - i\beta$ is also a solution.

If λ is a complex eigenvector for A, then so is $\bar{\lambda}$.

Prop Let A be a real $n \times n$ matrix w/ complex eigenvalue $\lambda = \alpha + i\beta$. Then $\bar{\lambda} = \alpha - i\beta$ is an eigenvalue. If $v = \vec{x} + i\vec{y}$ is an eigenvector for λ, then $\bar{v} = \vec{x} - i\vec{y}$ is an eigenvector for $\bar{\lambda}$.

Ex If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\lambda = i \qquad \bar{\lambda} = -i$

$v = \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2$

$\lambda = 1, \lambda = 1$

$\ker(A - I) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If V is a vector space w/ real scalars.

$$V^* = \mathrm{Hom}(V, \mathbb{R})$$
$$= \{\text{all linear functions } V \longrightarrow \mathbb{R}\}$$

This the definition

V^* is also a vector space.

If V has a basis $\{v_1 \ldots v_n\}$

what is a basis of V^*?

$f(x) = 3\cos x + 2\sin x$ is a linear comb of $\cos x$ $\sin x$

Given a basis $v_1 \ldots v_n$ of V

make a basis of V^*

$v_1^* \; v_2^* \ldots v_n^* \in V^*$

$$\ell_1 = v_1^* : V \longrightarrow \mathbb{R}$$
$$\ell_2 = v_2^* : V \longrightarrow \mathbb{R}$$
$$\vdots \qquad\qquad \vdots$$
$$\ell_n = v_n^* : V \longrightarrow \mathbb{R}$$

$$\ell_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\ell_i(c_1 v_1 + \ldots + c_n v_n)$$
$$= c_1 \ell_i(v_1) + \ldots + c_i \ell_i(v_i) + \ldots + c_n \ell_i(v_n)$$

$$l_i(v) = c_1 \cancel{l_i(v_1)}^{0} + \dots + c_i \underbrace{l_i(v_i)}_{1} + \dots + c_n \cancel{l_i(v_n)}^{0}$$

$$l_i(v_j) = \begin{cases} 1 & i \neq j \\ 0 & \text{otherwise} \end{cases}$$

$$= c_1 \cdot 0 + \dots + c_i \cdot 1 + \dots + c_n \cdot 0$$

$$l_i(v) = c_i$$

l_i spits out the coefficient of v_i.

$l_1 \dots l_n$ form a basis of V^* $= \{\text{all linear functions}\}$

Ex $V = \mathbb{R}^n$ $e_1 \dots e_n$

$e_i^* : \mathbb{R}^n \to \mathbb{R}$ $\quad e_i^*(a_1, \dots, a_n) = a_i$.

e_i^* form a basis of $(\mathbb{R}^n)^*$

$e_i^* = (0 \dots 1 \dots 0)$ $\quad$ = row vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_1^* \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \text{coefficient of } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$a_1 v_1 + a_2 v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{7}{2} \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \boxed{-\frac{1}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_1^* \begin{pmatrix} 3 \\ 4 \end{pmatrix} = -\frac{1}{2} \qquad v_2^* \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{7}{2}$$

$$(V^*)^* : \left\{ \begin{array}{c} \text{linear function} \\ V^* \to \mathbb{R} \end{array} \right\}$$

$$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2) = \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$$

$= $ vector space of all linear functions $\mathbb{R}^2 \to \mathbb{R}^2$

Are the functions such that

$\underline{L}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ a subspace of $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$?

$\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) = 2 \times 2$ matrices

$$L\begin{pmatrix}a\\b\end{pmatrix} = A\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a = 0$$

$$c = 0$$

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \operatorname{span}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{dx} : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$$A \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ \vdots \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} 0 & 0 & \cdots & & \\ 1 & 0 & \cdots & & \\ & 2 & 0 & & \\ & & 3 & 0 & \cdots \\ & & \vdots & & \end{pmatrix}$$

$u'' - 9u = x + \sin x$

① Find a particular solution to

$u'' - 9u = x \quad \alpha(x)$

② Find a particular solution to

$u'' - 9u = \sin x \quad \beta$

③ Find all solutions to

$\left.\begin{array}{l} \\ u'' - 9u = 0 \end{array}\right\} \rightarrow$ Find the kernel

$\frac{\partial^2}{\partial x^2} + \frac{\partial^0}{\partial x^0}$

$u = e^{rx}$ solve for r. $r = \pm 3$

$f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} +$

$f(x) = \underline{c_1 e^{3x} + c_2 e^{-3x}} + \underbrace{\alpha(x)}_{u_1^*} + \underbrace{\beta(x)}_{u_2^*}$

$u'' - 9u = x + \sin x$

The general solution is a formula to all solutions to this diff eq.

$$\underbrace{u(x)}_{\text{all solutions to } u''-9u = x+\sin x} = \underbrace{c_1 e^{-3x} + c_2 e^{3x}}_{\text{homogeneous solution}} + \underbrace{u_1^* + u_2^*}_{\text{particular solutions}}$$

homogeneous solution $\Updownarrow$ all solutions to $u''-9u=0$

u_1^* is 1 solution to $u''-9u=x$

u_2^* any 1 solution to $u''-9u=\sin x$