Yesterday... defined eigenvalues / eigenvectors

$Av = \lambda v$

subspace of all eigenvectors for λ.

$\rightarrow \quad V_\lambda = \ker(A - \lambda I)$

If λ is an eigenvalue

V_λ is a nontrivial subspace.

V_λ is sometimes called the <u>eigenspace</u> of λ.

Two difficulties

(1) If λ repeats, there may be too few eigenvectors.

alg. mult of $\lambda \geq \dim V_\lambda$.

# of times λ repeats $\geq$ # of ind. eigenvectors

② $\det(A - \lambda I) = 0$ may have complex solutions.

$A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ as complex vector space

You may not be able to find λ, V_λ if you are over the reals,

E.g. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

has no real eigenvalues / eigenvectors.

$\det(A - \lambda I) = \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i \qquad v_i = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$v_{-i} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ has eigenstuff

Prop A has an eigenvalue $\lambda = 0$ iff A is not invertible.

Pf: A has eigenvalue $\lambda = 0$

$\iff \exists v \neq 0$ s.t.

$Av = 0 \cdot v = 0$

$\iff v \in \ker(A)$

$\iff \ker(A) \neq 0$

$\iff$ not invertible (by big thm from Ch. 2)

If you find that $\lambda = 0$ is an eigenvalue, then A was not invertible.

$$\lambda = 0 : V_0 = \{ v \in \mathbb{R}^n \mid Av = 0v \} = \ker(A)$$

Recall: If A is a square matrix we define trace $\text{tr}(A)$ to be

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

Ex

$$\text{tr}\begin{pmatrix} \boxed{1} & 0 & 1 \\ -1 & \boxed{2} & 1 \\ 3 & 1 & \boxed{2} \end{pmatrix} = 1 + 2 + 2 = 5$$

Thm Let A be $n \times n$ w/ eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_n$ (may repeat).

Then

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$$

and

$$\text{tr}(A) = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Ex

$$A = \begin{pmatrix} \underline{0} & -1 & -1 \\ 1 & \underline{2} & 1 \\ 1 & 1 & \underline{2} \end{pmatrix} \qquad \lambda = 1, \lambda = 1, \lambda = 2$$

$$\det(A) = 1 \cdot 1 \cdot 2 = 2$$

$$\text{tr}(A) = 0 + 2 + 2 = 4$$
$$= 1 + 1 + 2 = 4$$

Pf: Since $P_A(\lambda) = \det(A - \lambda I)$

is a polynomial then, it is of the

form

$$P_A(\lambda) = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1 \lambda^1 + c_0$$

Let's use the properties of $\det(A - \lambda I)$

to calculate c_n, c_{n-1}, c_0.

Claim: $c_n = (-1)^n$.

The only way to get λ^n is by

multiplying the diagonals of $A - \lambda I$.

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & \\ a_{21} & a_{22} - \lambda & & \\ \vdots & & \ddots & \\ \vdots & \cdots & & a_{nn} - \lambda \end{pmatrix}$$

So $\det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & \ddots & \\ \vdots & & a_{nn}-\lambda \end{pmatrix}$

the only way to get λ^n is by looking at

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

By FOILing,

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$$= (-1)^n\lambda^n + \underline{(a_{11}+a_{22}+\cdots+a_{nn})}(-1)^{n-1}\lambda^{n-1} + \cdots$$

Since this is the only term in the determinant that contributes to λ^n.

the leading coefficient of

$$p_A(\lambda) = \det(A - \lambda I)$$

is $(-1)^n$.

<u>Eg</u>. $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\underbrace{(3-\lambda)(3-\lambda)}_{\text{only term that contributes to } \lambda^2} - 1$$

$$\underbrace{(-1)^2}\lambda^2 + (3+3)(-1)^1\lambda + 9 - 1$$

$$\boxed{1} \cdot \lambda^2 - 6\lambda + 8$$

Eg $\quad A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$p_A(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} 0-\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$= \underbrace{(0-\lambda)(2-\lambda)(2-\lambda)}_{\text{only term that contributes to } \lambda^3} + \cdots$$

$$= (-1)^3\lambda^3 + (0+2+2)(-1)^2\lambda^2$$

$$+ \cdots$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

If $P_A(\lambda) = c_n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$

$c_n = (-1)^n$

$P_A(\lambda) = (-1)^n \lambda^n + \boxed{c_{n-1}} \lambda^{n-1} + \dots + c_1 \lambda + c_0$

Turns out that the only term that contributes to λ^{n-1} is also

$(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$

$$\det\begin{pmatrix} 0-\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = \underline{(0-\lambda)} \det\begin{vmatrix} \underline{2-\lambda} & 1 \\ 1 & \underline{2-\lambda} \end{vmatrix}$$
$$- (-1)\det\begin{pmatrix} 1 & 1 \\ 1 & 2-\lambda \end{pmatrix}$$
$$+ (-1)\det\begin{pmatrix} 1 & 2-\lambda \\ 1 & 1 \end{pmatrix}$$

only way to get λ^3

only way to get λ^2

Therefore

$$P_A(\lambda) = (-1)^n \lambda^n + \underbrace{c_{n-1}\lambda^{n-1}} + \dots + c_1\lambda + c_0$$

$$= (-1)^n \lambda^n + \underbrace{(a_{11} + \dots + a_{nn})(-1)^{n-1}} \lambda^{n-1}$$

$$+ \dots + c_1\lambda^1 + c_0$$

FOILing tells us that

$$c_{n-1} = (a_{11} + \dots + a_{nn})(-1)^{n-1}$$

Eg $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$$P_A(\lambda) = \lambda^2 - 6\lambda + 8$$

$$-6 = (-1)^{2-1}(3+3)$$

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$P_A(\lambda) = -\lambda^3 + \underline{4}\lambda^2 - 5\lambda + 2$$

$$4 = (-1)^{3-1}(0+2+2)$$

FOILing

$$p_A(\lambda) = (-1)^n \lambda^n + \underbrace{(-1)^{n-1}(a_{11} + \dots + a_{nn})}\lambda^{n-1}$$

$$+ \dots + c_1 \lambda + c_0 .$$

On the other hand ... $\lambda_1, \dots, \lambda_n$

are solutions to $p_A(\lambda)$.

$$p_A(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$

FOIL ...

$$= (-1)^n \lambda^n + (-1)^n (-1)(\lambda_1 + \lambda_2 + \dots + \lambda_n)\lambda^{n-1}$$

$$+ \dots + (-1)^n (-1)^n (\lambda_1 \dots \lambda_n)$$

$$= (-1)^n \lambda^n + \underbrace{(-1)^{2n-1}(\lambda_1 + \dots + \lambda_n)}\lambda^{n-1}$$

$$+ \dots + \boxed{\lambda_1 \lambda_2 \dots \lambda_n}$$

So $\quad c_{n-1} = (-1)^{n-1}(a_{11} + \dots + a_{nn})$

$c_{n-1} = (-1)^{n+1}(\lambda_1 + \dots + \lambda_n)$

$$\Rightarrow \cancel{(-1)^{n+1}}(a_{11} + \dots + a_{nn})$$

$$= \cancel{(-1)^{n+1}}(\lambda_1 + \dots + \lambda_n)$$

$$\Rightarrow \underline{\lambda_1 + \dots + \lambda_n} = \underline{a_{11} + \dots + a_{nn}}$$

$$= tr(A).$$

$c_0 = \lambda_1 \lambda_2 \cdots \lambda_n.$

But $\quad c_0 = P_A(0)$

$$= (-1)^n 0^n + c_{n-1} 0^{n-1} + \dots + c_1 0 + c_0$$

$P_A(0) = \det(A - 0I) = \det(A)$

$\underline{\det(A)} = \underline{\lambda_1 \lambda_2 \cdots \lambda_n}$ $\quad \square$

Revisit this result:

alg mult. of $\lambda \geq \dim(V_\lambda)$

\# of times λ repeats $\geq$ \# of ind. eigenvectors

Eg $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\lambda = 1, \lambda = 1, \lambda = 2$

alg mult of $\lambda = 1$ is 2.

$\dim(V_{\lambda=1}) = \dim(\ker(A - 1I))$

$= \dim \ker \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$= 2$

alg mult of $\lambda = 1 = \dim V_1$.

Eg $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 = 0$$

$\lambda = 1, \lambda = 1$ alg mult of $\lambda = 1$
$= 2$

$\dim(V_1)$

$$= \dim(\ker(A - 1I))$$

$$= \dim\left(\ker\begin{array}{cc} \text{free} & \\ \left(\begin{matrix} \boxed{\begin{matrix}0\\0\end{matrix}} & \begin{matrix}1\\0\end{matrix}\end{matrix}\right) & \end{array}\right) = \text{\# of free columns}$$

$$= 1$$

alg mult $= \boxed{2}$ $\dim(V_1) = \boxed{1}$

Even though $\lambda = 1$ repeats, we get 1 eigenvector.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x$$

So $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda = 1, \lambda = 1$

When do we get the same number of independent eigenvectors as # of eigenvalues up to repetition?

Not always!!

When do we have enough eigenvectors?

Diagonalization ...

Prop Let $\lambda_1 \dots \lambda_k$ be <u>distinct</u> $(\lambda_i \neq \lambda_j)$ (no <u>repeats</u>) eigenvalues. Let $v_1 \dots v_k$ be a choice of associated eigenvectors.

Then $v_1 \dots v_k$ are independent.

Pf: Let

$$c_1 v_1 + \dots + c_k v_k = 0.$$

$$A(c_1 v_1 + \dots + c_k v_k) = 0$$

$$c_1 A v_1 + \dots + c_k A v_k = 0$$

$$c_1 \lambda_1 v_1 + \dots + c_k \lambda_k v_k = 0$$

since $v_1 \dots v_k$ are eigenvectors!

$$\lambda_k (c_1 v_1 + \dots + c_k v_k) = 0$$

$$c_1 \lambda_k v_1 + \dots + c_k \lambda_k v_k = 0$$

Take 2 equations

$$c_1 \lambda_1 v_1 + \dots + \cancel{c_k \lambda_k v_k} = 0$$

$$c_1 \lambda_k v_1 + \dots + \cancel{c_k \lambda_k v_k} = 0$$

Subtract --

$$c_1(\lambda_1 - \lambda_k) v_1 + c_2(\lambda_2 - \lambda_k) v_2 + \dots + c_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1} = 0$$

$\longrightarrow$ done by induction

We reduced to showing $v_1 \dots v_{k-1}$ are independent.

$\vdots$

Repeat process to reduce $v_1 \dots v_{k-2}$

Repeat until ... $\{v_1\}$, which is independent $\Rightarrow c_1 = 0$

Plug back into previous step $\Rightarrow c_2 = 0$

$\dots \quad c_n = 0$

If $c_1 \dots c_{k-1} = 0$

$$\Rightarrow \cancel{c_1}^{0} v_1 + \dots + c_k v_k = 0$$

$$c_k v_k = 0.$$

v_k is an eigenvector so
$v_k \neq 0$

$$\Rightarrow c_k = 0. \qquad \square$$

Induction ...

Distinct eigenvalues have independent eigenvectors!

<u>Thm</u> If $\lambda_1 \dots \lambda_n$ are distinct real eigenvalues of a $n \times n$ matrix w/ eigenvectors $v_1 \dots v_n$, then

$$\{v_1 \dots v_n\} \text{ is a basis of } \mathbb{R}^n.$$

$\square$

Ex $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

$\lambda = 2, \ \lambda = 4$

these eigenvalues are distinct!

No repeats

$\Longrightarrow v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

form a basis of $\mathbb{R}^2$.

Partial result

if λ has no repeats $\Longrightarrow$ eigenvectors form a basis!

Change of basis formula.

$T(x) = Ax$ in standard coords

$T(x) = Bx$ in $\{v_1, v_2\}$ coords

$B = S^{-1}AS \quad S = (v_1\, v_2)$.

What if our basis is the basis of eigenvectors?

Ex $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x) = Ax$ in standard coordinates.

But what if we picked

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ coords?

$$T\begin{pmatrix} x \\ y \end{pmatrix}_{v_1 v_2}$$

$$= T\left(x\begin{pmatrix} -1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$= xT\begin{pmatrix} -1 \\ 1 \end{pmatrix} + yT\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$ $\lambda = 4$

$$= xA\begin{pmatrix} -1 \\ 1 \end{pmatrix} + yA\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= x\,2\begin{pmatrix} -1 \\ 1 \end{pmatrix} + y\,4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (2x)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + (4y)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \end{pmatrix}_{v_1 v_2}$$

In the coordinates of a basis of eigenvectors

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 4y \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is diagonal!

<u>Def</u> A matrix A is <u>diagonalizable</u> if there exists a invertible matrix S such that $\Lambda = S^{-1}AS$ is diagonal.

A matrix "A" is diagonalizable if A is diagonal in a different basis.

Ex $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ is diagonalizable!

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

(change of basis formula)

$S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ columns are the basis of eigenvectors.

In general the way to diagonalize a matrix is to change basis to basis of eigenvectors, should one exist!

Ex $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable

the only eigenvector is $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

no basis to change to.

$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Thm A matrix $\underline{A}$ is diagonalizable iff the eigenvectors of A form a basis of $\mathbb{C}^n$.

possibly complex eigenvectors...

If λ are real, you can replace w/ $\mathbb{R}^n$.

Pf: Assume we have eigenvectors $\lambda_1 \dots \lambda_n$ and $v_1 \dots v_n$ form a basis.

Then $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

here we need that $v_1 \dots v_n$ form a basis

$= (v_1 \dots v_n)^{-1} A (v_1 \dots v_n)$

$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ is A but in coordinates $v_1 \dots v_n$.

In $v_1 \dots v_n$ coordinates

eigenvectors

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_v$$

$$= A(x_1 v_1 + \dots + x_n v_n)$$

$$= x_1 A v_1 + \dots + x_n A v_n$$

$$= x_1 \lambda_1 v_1 + \dots + x_n \lambda_n v_n$$

$$= \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix}_v$$

$$\Longrightarrow \quad T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_v = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_v$$

$$\Longrightarrow \quad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = (v_1 \dots v_n)^{-1} A (v_1 \dots v_n)$$

by change of basis.

Assume $\Lambda = S^{-1}AS$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Claim: $\lambda_1 \dots \lambda_n$ are the eigenvalues of A
columns of S are the eigenvectors.

Pf Since $\Lambda = S^{-1}AS$

$$\Rightarrow AS = S\Lambda$$

$$\Rightarrow \underline{(AS)_k} = \underline{(S\Lambda)_k} \quad (k^{th} \text{ column})$$

If $S = (v_1 \dots v_n)$

then $AS = (Av_1 \dots Av_n)$

$$(AS)_k = \underline{Av_k}$$

$$(S\Lambda)_k = S(\Lambda)_k = S\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{\lambda_k v_k}$$

$\Longrightarrow \quad A v_k = \lambda_k v_k$

$\Longrightarrow \quad \lambda_1 \ldots \lambda_n$ are eigenvalues

$\Longrightarrow$ columns of S are a basis of eigenvectors.

Ex $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\lambda = 1, \lambda = 1, \lambda = 2$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a basis of eigenvectors

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

is the diagonalization!

§8.4

Def Let $T : \underline{V} \to \underline{V}$ be a linear transformation.

We say a subspace $W \subseteq V$ is __invariant__ if $\forall w \in W$

$T(w) \in W \subseteq V$.

Ex Let A be an $n \times n$ matrix.

$$A : \underbrace{\mathbb{R}^n}_{V} \longrightarrow \underbrace{\mathbb{R}^n}_{V}$$

Claim: If λ is an eigenvalue then V_λ is invariant under A.

If $w \in V_\lambda$, then $Aw = \lambda w$.

$\Rightarrow$ But $\lambda w \in V_\lambda$ (closed under scalar mult.)

$\Rightarrow Aw \in V_\lambda$, V_λ is invariant.

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ Finding

invariant subspaces is the same

as finding eigenspaces.

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Suppose $W = \operatorname{span}(a,b)$ is

invariant.

$\forall w \in W, \quad T(w) \in W$.

$w = c(a,b) \in \operatorname{span}\begin{pmatrix} a \\ b \end{pmatrix}$

If $T(w) \in W$

$$\Rightarrow \quad T\left(c\begin{pmatrix} a \\ b \end{pmatrix}\right) = cT\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\in \operatorname{span}\begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow T\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{span}\begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix}$ is a eigenvector for T.

Ex

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

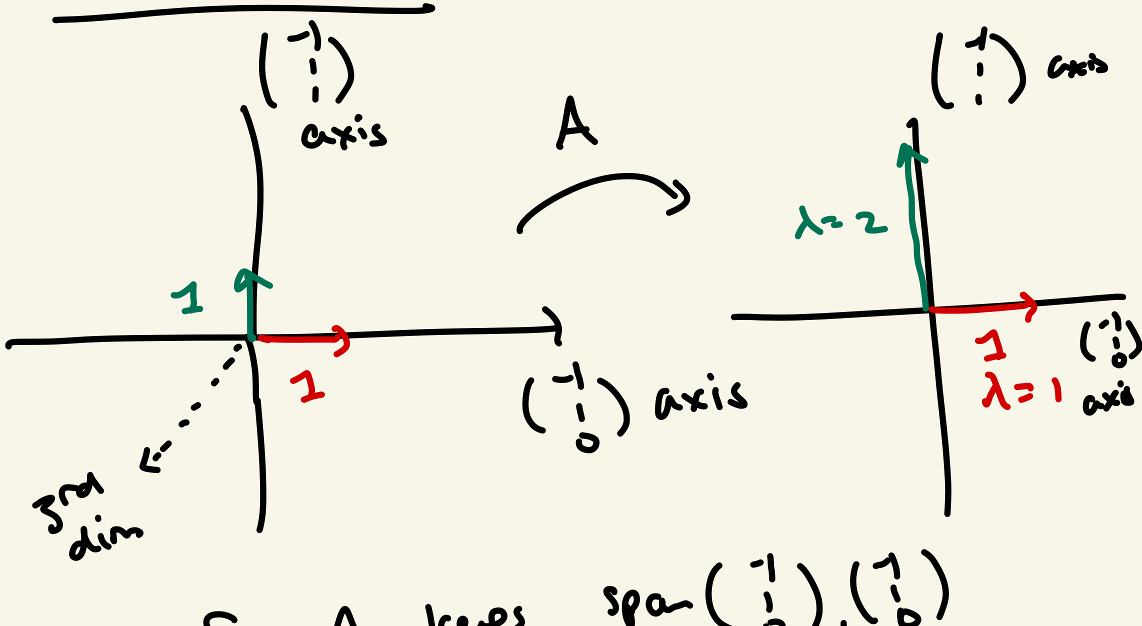
$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\lambda = 1$

$v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\lambda = 1$

$v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $\lambda = 2$

span$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$ is invariant!

span$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right)$ is invariant!



So A keeps span$\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$ invariant.

When do the eigenvectors form a basis of $\mathbb{R}^n$?

We had enough eigenvectors

when
$\dim(V_\lambda) = \underline{\text{alg. mult of } \lambda}$

If these are $=$ for λ, then we have a basis.

If alg mult $\lambda = 1$, aka λ_i are distinct

then $v_1 \dots v_n$ was a basis.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\dim V_1 <$ alg mult of $\lambda = 1$

$\Downarrow$

no basis!

Same as diagonalizing!

When is a matrix diagonalizable?

Thm Let A be a symmetric real matrix.

(a) All eigenvalues of A are real.

(b) Eigenvectors to distinct eigenvalues are orthogonal.

(c) There is an orthonormal basis of eigenvectors of $\mathbb{R}^n$.

(d) All symmetric matrices are diagonalizable in $\mathbb{R}^n$ by an orthogonal matrix. (Just a–c summarized)

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\to T = \begin{pmatrix}0 & b\\0 & d\end{pmatrix}$$

$$T = b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$W = \operatorname{span}\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right)$$

$$\dim W = 2$$

$$(\mathbb{R}^n)^* = \left.\begin{matrix}\text{All linear functions}\\ \mathbb{R}^n \to \mathbb{R}^1\end{matrix}\right\}$$

$= 1 \times n$ matrices

$=$ row vector

$\underline{l_i} \longleftrightarrow$ row matrix ? rows of A^{-1}

$l_i \longleftrightarrow i^{th}$ row of A^{-1}

$\underline{l_i}$ $(c_1 v_1 + \dots + c_n v_n) = c_i$

$l_i\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = ??$

In general if $T(v_1) = b_1$

$T(v_2) = b_2$

$\vdots$

$T(v_n) = b_n$

$$A = (b_1 \dots b_n).$$

$$A_{l_i} = (l_i(v_1) \;\; l_i(v_2) \dots \; l_i(v_n))$$

$$= \begin{array}{cccccc} & & & i & & \\ (0 & 0 & \dots & 1 & \dots & 0) \end{array}$$

$l_i \rightsquigarrow e_i^T$ in $v_1 \dots v_n$ coordinates

what is e_i^T in standard coordinates?

$$l_i(\vec{x}) = l_i(x_1 e_1 + \dots + x_n e_n)$$

$$= x_1 l_i(e_1) + \dots + x_n l_i(e_n)$$

If we know $l_i(e_1), \dots, l_i(e_n)$

$$\begin{pmatrix} l_i(e_1) & \dots & l_i(e_n) \end{pmatrix} = i^{th} \text{ row of } A^{-1}.$$

$l_i(e_j) = ??$

How to write $e_j = c_1 v_1 + \dots + c_n v_n$?
what's c_j?

$$Ac = \begin{pmatrix} v_1 \dots v_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} j$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = A^{-1} e_j$$

$\Rightarrow c_j = j^{th}$ row of $A^{-1} \cdot e_j$

(b) $\{l_i\}$ is a basis for all linear functions $V \longrightarrow \mathbb{R}$

$$T = c_1 l_1 + \dots + c_n l_n$$

$$T(x) = c_1 l_1(x) + \dots + c_n l_n(x) \quad \forall x$$

If $\underline{c_1 l_1 + \dots + c_n l_n = 0}$ as functions

WTS $c_i = 0$.

$$(c_1 l_1 + \dots + c_n l_n)(v_1) = 0$$

$$c_1 \overset{1}{\cancel{l_1(v_1)}} + c_2 \overset{0}{\cancel{l_2(v_1)}} + \dots + c_n \overset{0}{\cancel{l_n(v_1)}} = 0$$

$$c_1 = 0$$

$$(c_1 l_1 + \dots + c_n l_n)(v_2) = 0$$

$$\Rightarrow c_2 = 0$$

etc.

So independent?

Span Let $T : V \to \mathbb{R}$

$T(v_1) = d_1$ $*$

$T(v_2) = d_2$

$\vdots$

$T(v_n) = d_n$

Claim

$\underline{T = d_1 \ell_1 + \dots + d_n \ell_n}$

To show T and $d_1 \ell_1 + \dots + d_n \ell_n$ are equal as functions.

we need to show

$$T(v) = (d_1 \ell_1 + \dots + d_n \ell_n)(v) \quad \forall v \in V.$$

$$= d_1 \ell_1(v) + \dots + d_n \ell_n(v)$$

It suffices by linear to show that

$$T(v_i) = (d_1 \ell_1 + \dots + d_n \ell_n)(v_i)$$

for all basis vectors v_i.

If S, T agree on $v_1 \ldots v_n$ then $S = T$.

$$S(v) = S(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 S(v_1) + \cdots + c_n S(v_n)$$

$$= c_1 T(v_1) + \cdots + c_n T(v_n)$$

$$= T(c_1 v_1 + \cdots + c_n v_n) = T(v)$$

So $S = T$.

If 2 linear functions agree on a basis, they're the same linear function.

Apply $S = d_1 l_1 + \cdots + d_n l_n$.

$\underline{T(v_i) = d_i}$ by definition *

$\underline{(d_1\ell_1 + \dots + d_n\ell_n)}(v_i)$

$= d_1\cancelto{0}{\ell_1(v_i)} + \dots + d_i\cancelto{1}{\ell_i(v_i)} + \dots +$
$d_n\cancelto{0}{\ell_n(v_i)}$

$= d_i$ also. $\square$

$T \in \operatorname{span}(\ell_1, \dots, \ell_n)$

To show $T \in \operatorname{span}(\ell_1 \dots \ell_n)$

$\underline{T = c_1\ell_1 + \dots + c_n\ell_n}$ what are $c_1 \dots c_n$?

$c_1 = T(v_1) = d_1$

$c_2 = T(v_2) = d_2$

$\vdots$

$c_n = T(v_n) = d_n$