When does a matrix have enough eigenvectors to form a basis?

$\Updownarrow$

when is a matrix diagonalizable?

Thm Let A be a real symmetric matrix.

(a) All eigenvalues are real. ✓

(b) Eigenvectors for distinct eigenvalues are orthogonal. ✓

(c) There is an orthonormal basis of eigenvectors of A for $\mathbb{R}^n$. ✓

(d) All symmetric matrices are diagonalizable by an orthogonal matrix in $\mathbb{R}^n$. ✓

Pf : (a) Since A is real and symmetric, $\bar{A} = A$, and $A^T = A$.

So all real matrices have possibly complex eigenvalues / eigenvectors

$A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, so we consider complex scalars and complex dot product.

$$\left(z \cdot w = z^T \bar{w} \right)$$

Claim : $Av \cdot w = v \cdot Aw \quad \forall v, w$

Pf $Av \cdot w = (Av)^T \bar{w} = v^T A^T \bar{w}$

$$= v^T \bar{A} \bar{w} = v^T (\overline{Aw})$$

$$= v \cdot Aw$$

We can use this formula to λ is real.

Let λ be an eigenvalue of A, let v be a corresponding eigenvector.

Then

$$Av \cdot v = \lambda v \cdot v = \lambda \|v\|^2$$

$$Av \cdot v \overset{*}{=} v \cdot Av = v \cdot (\lambda v)$$

$$= \bar{\lambda}\, v \cdot v = \bar{\lambda} \|v\|^2$$

$$\implies \lambda \|v\|^2 = \bar{\lambda} \|v\|^2$$

In fact $\|v\|^2 \neq 0$ because v is an eigenvector, $v \neq 0$.

$$\implies \lambda = \bar{\lambda} \implies \lambda \in \mathbb{R}.$$

So all eigenvalues of symmetric matrices are real!

(b) Let λ, μ be distinct eigenvalues.

Let $v \in V_\lambda$, $w \in V_\mu$ $v, w \neq 0$.

WTS $v \cdot w = 0$.

Consider $Av \cdot w$. By part (a), we no longer have any reason to consider complex scalars. λ, μ, v, w are all real. We're back to the real dot product.

$$Av \cdot w = \lambda v \cdot w$$

$$Av\, w \overset{* !!}{=} v \cdot Aw = v \cdot (\mu w)$$

$$= \mu\, v \cdot w$$

$$\lambda (v \cdot w) = \mu (v \cdot w)$$

We know that $\lambda \neq \mu$.

$\lambda - \mu \neq 0$.

$$\Longrightarrow \quad (\lambda-\mu)(v\cdot w) = 0 \qquad \lambda-\mu \neq 0$$

$$v\cdot w = 0 .$$

So eigenvectors for distinct eigenvalues are orthogonal.

(c) We need to show that $\mathbb{R}^n$ has an orthonormal basis of eigenvecs of A.

So far, (b) says that $\lambda \neq \mu$

$$V_\lambda \perp V_\mu .$$

Oh, since V_λ are orthogonal to each other, then you can find a basis of V_λ and G-S each one individually. Then you'll get an ortho~~normal~~ basis.

only mutually orth.

If $v_{1,\lambda} \dots v_{k,\lambda} \in V_\lambda$

$w_{1,\mu} \dots, w_{r,\mu} \in V_\mu$

by G-S $v_{i,\lambda} \cdot v_{j,\lambda} = 0$, $w_{i,\mu} \cdot w_{j,\mu} = 0$

by (b) $v_{i,\lambda} \cdot w_{j,\mu} = 0$

All together $\{v_1 \dots v_k, w_1 \dots w_r\}$ are mutually orthogonal!

But doesn't say that this set spans...

So it suffices to show that A is diagonalizable, then G-S on all the V_λ individually. will get you an orthonormal basis of eigenvectors.

Induction proof..

Let v be an eigenvector for λ.

Let $\underline{W = \text{span}(v)^{\perp}}$.

Claim: W is an invariant subspace for A.

- If $w \in W$, then $Aw \in W$. (definition of invariant yesterday---)

Let $w \in W$, $\underline{w \cdot v = 0}$.

Then $\underline{Aw \cdot v} \underset{*}{=} w \cdot Av$

$$= w \cdot \lambda v = \lambda (w \cdot v)$$

$$= \lambda \cdot 0 = \underline{0}$$

So $Aw \in \text{span}(v)^{\perp} = W$.

So $\text{span}(v)^{\perp}$ is invariant under A.

By rank-nullity

$$\overset{1}{\cancel{\dim(\text{span}(v))}} + \dim(\text{span}(v)^{\perp}) = n = \dim(\mathbb{R}^n).$$

$$1 + \dim(\text{span}(v)^{\perp}) = n$$

$$\dim(\text{span}(v)^{\perp}) = n-1.$$

$$\dim(W) = n-1$$

W is invariant and $\dim(W) = n-1$
$\dim(\mathbb{R}^n) = n$

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ but since W is invariant

$$A\big|_W : W \longrightarrow W$$

restrict domain to W

codomain is W since it's invariant

$$\begin{tikzcd}
A : & \mathbb{R}^n \arrow[r, "A \text{ symm}"'] \arrow[d, squiggly] & \mathbb{R}^n & \dim(\mathbb{R}^n) = n & u_1 \ldots u_n \\
A : & W \arrow[r, "A \text{ symm}"'] \arrow[dd, squiggly] & W & \dim(W) = n-1 & \\
 & (\text{Induction on } \dim(\mathbb{R}^n) = n) & & & u_1 \ldots u_{n-1} \arrow[lu] \\
A : & W' \arrow[r] \arrow[d] & W' & \dim = n-2 & u_1 \ldots u_{n-2} \arrow[u] \\
 & \vdots \arrow[dd] & & & \vdots \arrow[u] \\
 & & & & u_1, u_2 \arrow[u] \\
A : & \mathbb{R}^1 \arrow[r] & \mathbb{R}^1 & \dim = 1 & u \arrow[u, "(b)"']
\end{tikzcd}$$

So $A|_W : W \longrightarrow W$

by recursion, we can

make $\{u_1 \dots u_{n-1}\}$ an orthonorm.

basis of eigenvectors in $\text{span}(v)^\perp$

$\longrightarrow$

$$\left\{u_1, \dots, u_{n-1}, \frac{v}{\|v\|}\right\}$$

is an orthonormal basis of

eigenvectors on $\mathbb{R}^n$

(d)

Since $\{u_1 \dots u_n\}$ is a orth. basis

of eigenvectors, we can diagonalize

by a orthog matrix

$$Q = (u_1 \dots u_n) \longrightarrow Q^T = Q^{-1}$$

Q orthogonal

$$A = Q \Lambda Q^{-1} = Q \Lambda Q^T. \quad \square$$

Ex Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

$$\begin{aligned} P_A(\lambda) &= (3-\lambda)^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

$\lambda = 4, 2$ (a) says that these are real, which they are.

$\lambda = 4$

$A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$V_4 = \ker(A - 4I) = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *

$V_2 = \ker(A - 2I) = \ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$= \operatorname{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ *

In fact $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$ (b) predicted this

(c) $\mathbb{R}^2$ should have an orthonormal basis of eigenvectors

G-S V_2, V_4 individually

$$\begin{tikzcd}
V_4 \arrow[d] & V_2 \arrow[d] \\
\operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \arrow[d, "G-S"] & \operatorname{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \arrow[d, "G-S"] \\
\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}
\end{tikzcd}$$

part (c)

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ diagonalizing

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Called the spectral decomposition

$$\underset{\text{new coord}}{B} = S^{-1}AS$$

$$S = (v_1 \dots v_n)$$

$$A = SBS^{-1}$$

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{in } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

coordinates

If A, the adjoint A^*

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

is the unique matrix

s.t.

$$\langle Av, w \rangle = \langle v, A^* w \rangle \ \forall v, w$$

$$A : \underset{K}{\mathbb{R}^n} \longrightarrow \underset{L}{\mathbb{R}^m} \qquad \boxed{A^* = K^{-1}AL.}$$

$\langle v, w \rangle = 3v_1w_1 + 6v_2w_2$

$\longrightarrow \quad (v_1\ v_2)\, K \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} =$

If $K = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$

$(x\ y) \begin{pmatrix} \overset{x^2}{\underline{1}} & 2 \\ 2 & \underset{y^2}{\overline{1}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$x^2 + 4xy + y^2$

$\longrightarrow \quad (v_1\ v_2) \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$m \times m$	$m \times n$	$n \times n$
K	A	L

$P^2 = P$. let $v \in \text{img}(P)$.

then $Pv = v$.

$v \in \text{img}(P) \Rightarrow v = Pw$

$Pv = P \cdot Pw = P^2 w = Pw = v$

$Pv = Iv \quad \forall v \in \text{img}(P)$.

$N^k = 0$ for some k.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Unitary

Hermitian

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\underline{e^A} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots$$

$$\underline{e^{U}} = e^{D+N} = \underline{e^{D}e^{N}}$$

$$\begin{bmatrix} * & * & * \\ & * & * \\ & & * & * \end{bmatrix} = \begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} + \begin{bmatrix} 0 & & * \\ & 0 & \\ & & 0 \\ & & & 0 \end{bmatrix}$$

$$\underline{e^{D}} = e^{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$$

$$N^5 = 0$$

$$e^{N} = I + N + \frac{1}{2!}N^2 + \frac{1}{3!}N^3 + \frac{1}{4!}N^4 \}$$

$$+ \cancel{\frac{1}{5!}0} + \dots$$

$y_1 \cdots y_n$ $\qquad$ $y_i(t)$

$$y_1' = a_{11}y_1 + \cdots + a_{1n}y_1$$

$$\vdots$$

$$y_n' = a_{n1}y_1 + \cdots + a_{nn}y$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\begin{tikzcd} f' = af \arrow[d, bend left] \\ f = e^{ax} \end{tikzcd}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = e^{At}$$

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$$

$$\lim_{n \to \infty} (1-r)(1+r+\dots+r^n) = (1-r)\Sigma$$

$$\lim = 1 - r^{n+1} = 1$$

$$\Sigma = \frac{1}{1-r}$$

$$\chi : G \to M \xrightarrow{tr} \mathbb{R}$$

$$\sum_n \frac{x(n)}{n}$$ Dirichlet L-function

$\mathbb{Z} \Rightarrow \quad (1), (2), (3), (4) \ldots$

$\vdots$

prime ideals

$(2), (3), (5), (7)$

$\mathbb{Z}/(p) \Rightarrow$ simple

$(6) \ni 6 \qquad 2 \cdot 3 = 6$

$2; 3 \notin (6)$

max ideals of $\mathbb{C}$

$\Updownarrow$

elements of $\mathbb{C}$

$\mathbb{C}$ as a ring

$\mathbb{C}[x]$ is a ring

prime ideals $= \{ (f)$ f is an irreducible polynomial $\}$

$= \{ (x-a) \mid \} \Rightarrow a$

$\mathbb{C}[x,y]$

$\rightsquigarrow$ prime ideals

$$\Updownarrow$$

$x=a, y=b$

(x^2-y) Irreducible $f(x,y)$ and $\underline{(x-a, y-b)}$

$$\Updownarrow \qquad\qquad \Updownarrow$$

graph of $f(x,y)=0$ $\qquad$ $\underline{(a,b)}$

$\underline{x^2-y=0}$