Yesterday...

Every symmetric matrix is diagonalizable by an orthogonal matrix of eigenvectors!

This is also called the <u>spectral decomposition</u> of a symmetric matrix.

$$A = Q \Lambda Q^T \rightsquigarrow Q^T = Q^{-1}$$

$\hookrightarrow \Lambda$ diagonal matrix of λ_i.

Thm A symmetric matrix is positive definite **iff** all the eigenvalues are positive. $\Rightarrow$ $\Leftarrow$

Pf: Let K be positive definite. $\Rightarrow$

By def. $q(x) = x^T K x > 0$

$\forall \; x \neq 0$.

But K is symmetric, so

$K = Q \Lambda Q^T$, by spectral decomposition $\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$$q(x) = x^T K x = x^T Q \Lambda Q^T x$$

$$= (Q^T x)^T \Lambda (Q^T x)$$

$q(x) = (Q^T x)^T \Lambda (Q^T x)$

If we let $y = Q^T x$ $\quad$ Q^T is invertible

$\Rightarrow$ $\quad$ $x = Qy$

$q(y) = y^T \Lambda y$

$$q(x) = q(y) = (y_1 \dots y_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2 > 0$$

If $y = \vec{e_i}$ then $q(e_i) = \lambda_i > 0$

$\Leftarrow$ If $\lambda_i > 0$

$$q(x) = y^T \Lambda y = \sum \lambda_i y_i^2$$

Since $\lambda_i > 0$, $q(x) = \sum \lambda_i y_i^2 > 0$

$\square$

Ex $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\lambda = 2, \quad \lambda = 4$

4, 2 > 0 so A is positive definite!

$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\downarrow$ G-S $\qquad \downarrow$ G-S

$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To make an orthonormal basis of eigenvectors for A, do G-S on each V_λ individually.

Spectral decomposition

$$A = Q \Lambda Q^T$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \longrightarrow q(x) = x^T \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} x$$

$$= (x_1, x_2) \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 3x_1^2 + 2x_1x_2 + 3x_1^2$$

$$= 2x_1^2 + 2x_2^2 + x_1^2 + 2x_1x_2 + x_2^2$$

$$= 2x_1^2 + 2x_2^2 + (x_1 + x_2)^2 > 0$$

on the other hand

$$q(x) = (x_1, x_2) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} x_1 + x_2 & -x_1 + x_2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix}$$

let $y_1 = x_1 + x_2 \quad y_2 = -x_1 + x_2$

$$q(y) = \frac{1}{2} (y_1\ y_2) \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \frac{1}{2} \left(4y_1^2 + 2y_2^2 \right) > 0$$

$$q(x) = \frac{1}{2}\left(4y_1^2 + 2y_2^2\right) \qquad y_1 = x_1 + x_2$$
$$y_2 = -x_1 + x_2$$

$$= \frac{1}{2}\left(4(x_1+x_2)^2 + 2(-x_1+x_2)^2\right) > 0$$

$$= 3x_1^2 + 2x_1x_2 + 3x_2^2 > 0$$

Recall that

K being pos. def

$\Longleftrightarrow$ $\langle x, y\rangle = x^T K y$ defined an inner product

$f(x,y)$ min/max $\quad Df = 0$

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

if Hf pos def $\Longrightarrow$ min value

Ex Consider

$$q(x,y,z) = x^2 + 2xz + y^2 - 2yz \overset{?}{>} 0$$

write $q(x,y,z) = (x\ y\ z)\, K \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

and find the eigenvalues of K!

$$K = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \end{array} \qquad q(\vec{x}) = x^T \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} x$$

Compute eigenvalues $\longrightarrow$

$\boxed{\lambda = -1}$ not positive!

$\lambda = 1$

$\lambda = 2$

So $q(x) \not> 0$ for all x.

$\lambda = -1 \Longrightarrow v = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ eigenvector $\quad q(-1,1,2) < 0$

Thm let K be a symmetric matrix

let K_i = the matrix formed by the first i rows and columns of K.

(Ex $\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{matrix}$ $\quad K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Then TFAE

1) K is positive definite
2) all eigenvalues of $K > 0$
3) all pivots of K are positive
4) $\det K_i > 0 \quad \forall i.$ ✓

Pf 1) $\Leftrightarrow$ 2) ✓

$$\begin{tikzcd} & 1) \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ 3) \arrow[rr, Leftrightarrow] & & 4) \end{tikzcd}$$

equiv by $A = LDU$

What if you can't diagonalize?

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has no diagonalization

what can we say about it?

- Schur decomposition (not common)

- Jordan Canonical Form (Jordan decomposition) (more common)

more intuitively close to diagonalization

"more sophisticated"

Schur decomposition

If A is not symmetric, complex eigenvalues / eigenvectors are a possibility.

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^n, \text{ since}$$

$$v \in \mathbb{C}^n \text{ not } v \in \mathbb{R}^n$$

$$\text{or } \lambda \in \mathbb{C}.$$

$$z \cdot w = z^T \overline{w}$$

orthogonal matrices don't quite make sense in $\mathbb{C}^n$. We need a new concept, called a unitary matrix.

Def: A matrix $U \in M_{n\times n}(\mathbb{C})$ is called unitary if

$$U^{-1} = \overline{U^T}.$$

Def: $\overline{U^T}$ is often called the Hermitian of U.

$$U^H = \overline{U^T} \text{ OR } U^{+} = \overline{U^T} = \bar{U}^T$$

I've seen this

book, I'll use this one, NOT ADJOINT

Prop U is unitary iff the columns of U form an orthonormal basis of $\mathbb{C}^n$.

Pf: If U is unitary, then

$$U^{-1} = \overline{U^T}$$

$$U^{-1} U = I$$

But $U = (u_1 \; u_2 \cdots u_n)$

$u_i \in \mathbb{C}^n$

$$U^{-1}U = \begin{pmatrix} \overline{u_1} \\ \overline{u_2} \\ \vdots \\ \overline{u_n} \end{pmatrix} \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} = I$$

need

$$\Leftrightarrow \begin{cases} \overline{u_i} \cdot u_j = 0 & \text{if } i \neq j \\ \overline{u_i} \cdot u_i = 1 & \text{if } i = j \end{cases}$$

$\Leftrightarrow \{u\}_i$ form an orthonormal basis of $\mathbb{C}^n$!

$\square$

Ex $U = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = (ie_1 \quad ie_2)$

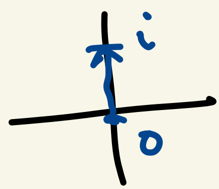
$$\overline{U^T} = \overline{\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$$

$$\overline{U^T}\, U = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} -i \cdot i & 0 \\ 0 & -i \cdot i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$U^T U = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \neq I$$

ie_1, ie_2 should be orthonormal



$$\|ie_1\| = |i|\,\|e_1\|$$
$$= 1 \cdot 1 = 1$$

If we want to generalize the spectral decomposition

$$A = \underbrace{\boxed{Q}}_{\text{orth.}} \Lambda Q^T$$

to a general matrix.

We need to possibly consider an orthogonal matrix but on $\mathbb{C}^n$

orth. + $\mathbb{C}^n \longrightarrow$ unitary

Prop If U_1, U_2 are unitary matrices, then so is $U_1 U_2$.

Pf: Same as for orth.

Thm (Schur Decomposition)

Let A be any $n \times n$ matrix.
Then there exists a unitary matrix U and upper triangular matrix Δ such that

$$A = U \Delta U^{\dagger} = U \Delta U^{-1}$$

and the diagonals of Δ are the eigenvalues of A.

orthogonal $\longrightarrow$ unitary

diagonal $\longrightarrow$ upper triangular

How to compute!

Take A, compute an eigenvalue $\lambda_1 \in \mathbb{C}$

eigenvector $v_1 \in \mathbb{C}^n$.

$$v_1 \longrightarrow \frac{v_1}{\|v_1\|} = u_1$$ also an eigen vector.

① Find a unitary matrix U_1 w/ u_1 as the first column.

$$(u_1\, v_2 \ldots v_n) \xrightarrow[G-S]{} (u_1\, \tilde{u}_2\, \tilde{u}_3 \ldots \tilde{u}_n)$$

$$U_1 = (u_1\, \tilde{u}_2\, \tilde{u}_3 \ldots \tilde{u}_n)$$

② Since u_1 is an eigenvector of A w/ eigenvalue λ_1.

$$U_1^{-1} A U_1 = \begin{pmatrix} \lambda_1 & * & * & * & * \\ 0 & * & * & * & * \\ \vdots & * & * & * & * \\ 0 & & \vdots & & \end{pmatrix}$$

$$U_1^{-1} A U_1 = \begin{pmatrix} \lambda_1 & * & * & * & * \\ 0 & * & * & * & * \\ \vdots & * & * & * & * \\ 0 & & \ddots & & \end{pmatrix}$$

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & r \\ \vec{0} & C \end{pmatrix},$$ this is a block matrix

λ_1 is 1×1 matrix

r is $1 \times n-1$

$\vec{0}$ is $n-1 \times 1$

C is $\boxed{n-1 \times n-1}$

We're actually done if we can Schur decompose C.

Assume $C = V \Gamma V^\dagger$, V is unitary, Γ upper triangular.

$\boxed{\text{recursive}}$

$$U_2 = \begin{bmatrix} 1 & \vec{0} \\ \vec{0} & V \end{bmatrix}$$ block matrix unitary

$U^\dagger = \overline{U^T}$

Claim!

$U_2^\dagger U_1^\dagger A \underbrace{U_1 U_2}_{U}$ is upper Δ if eigenvalues on diagonal

$$U_2^\dagger (U_1^\dagger A U_1) U_2 = U_2^\dagger \begin{pmatrix} \lambda_1 & r \\ 0 & C \end{pmatrix} U_2$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & V^\dagger \end{bmatrix} \begin{bmatrix} \lambda_1 & r \\ 0 & C \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & r \\ 0 & V^\dagger C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & rV \\ 0 & V^\dagger C V \end{bmatrix} = \begin{bmatrix} \lambda_1 & s \\ 0 & \Gamma \end{bmatrix} = \Delta$$

Γ is already upper triangular w/ $\lambda_2 \dots \lambda_n$ on diagonal!

$\square$

$$\begin{pmatrix} \lambda_1 & & \\ 0 & & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_1 & & \\ 0 & & \\ 0 & & C \\ 0 & & \\ 0 & & \end{pmatrix}$$

diagonalize

$$\longrightarrow \begin{pmatrix} \lambda_1 & \overset{x}{x} & x & x & x & x \\ 0 & \lambda_2 & x & x & x & x \\ 0 & 0 & & & & \\ 0 & 0 & & & D & \\ \vdots & 0 & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

$\longrightarrow$ go until done.

Ex Let $A = \begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix}$

Not a diagonalizable matrix

$$P_A(\lambda) = 2\lambda^2 - \lambda^3 = 0$$

$$= \lambda^2(2-\lambda) = 0$$

$$\underbrace{\lambda = 0, \lambda = 0}_{\text{alg mult.} = 2}, \lambda = 2$$

$$V_{\lambda=0} = \ker(A - 0I) = \ker(A)$$

$$= \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \underline{1\text{-D} < 2}$$

row reduce

not diagonalizable

$$V_{\lambda=2} = \ker(A - 2I) = \operatorname{span}\underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

Let's compute Schur decomposition of A

Let's pick $\lambda_1 = 2$, $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

" unit eigenvector.

① Find a unitary matrix w/ $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ as the first column.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow[\text{G-S}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\boxed{U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}}$$

diagonalizes first column only

$$\overline{U_1^T} A U_1 = \begin{pmatrix} 2 & -8 & \frac{\sqrt{5}}{2} \\ 0 & 2 & -\frac{1}{\sqrt{2}} \\ 0 & 4\sqrt{2} & -2 \end{pmatrix}$$

Take this and do this in the 2x2 case.

diagonalize this part

$$C = \begin{pmatrix} 2 & \frac{-1}{\sqrt{2}} \\ 4\sqrt{2} & -2 \end{pmatrix} \rightsquigarrow \boxed{\begin{matrix} \lambda = 0 \\ \lambda = 0 \end{matrix}}$$

But $V_0 = \ker(A - 0I) = \ker(A)$

$$= \operatorname{span}\begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}$$

Unit vector is $u_2 = \frac{1}{3}(1, 2\sqrt{2})$

$$u_2, e_2 \xrightarrow{G-S} \frac{1}{3}\begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2\sqrt{2} \\ -1 \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V = \boxed{\frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix}}$$

? Plug into previous step

$$VCV^T = \begin{pmatrix} 0 & \frac{4}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & -1/3 \end{pmatrix}$$

$$\underline{U_2^\dagger U_1^\dagger} A U_1 U_2 = \Delta$$

$$\underbrace{(U_1U_2)^T A \underbrace{U_1U_2}_{U} = \Delta}_{\text{upper } \Delta \text{ w/ } \lambda \text{ on diagonal}}$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}$$

$$A = U \Delta \overline{U^T}$$

$$\Delta = U_2^{\dagger}\,\underbrace{\left(U_1^{\dagger} A U_1\right)}\,U_2$$

$$= U_2^{\dagger} \begin{pmatrix} 2 & -8 & \frac{\sqrt{5}}{2} \\ 0 & 2 & -\frac{1}{\sqrt{2}} \\ 0 & 4\sqrt{2} & -2 \end{pmatrix} U_2^{\dagger}$$

$$= \begin{pmatrix} \boxed{2} & 2/3 & \frac{-37}{3\sqrt{2}} \\ 0 & \boxed{0} & 9/\sqrt{2} \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

eigenvalues on diagonal

$$A = U \Delta U^{T}$$

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix}$$

Jordan Canonical form

Thm Let A be a matrix, $n \times n$.

Then $A = SJS^{-1}$ where J is a matrix of the form

$$\begin{bmatrix} \underline{J_{\lambda_1, k_1}} & & & \\ & \underline{J_{\lambda_2, k_2}} & & \\ & & \ddots & \\ & & & \underline{J_{\lambda_n k_n}} \end{bmatrix}.$$

block matrix

What is $J_{\lambda, k}$ in general?

$J_{\lambda,k}$ is called a Jordan block.

a $k \times k$ matrix w/ λ on the diagonal and 1's on the superdiagonal (diagonal above).

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}$$

$k \times k$

$$J_{5,3} = \begin{bmatrix} 5 & 1 & \\ & 5 & 1 \\ & & 5 \end{bmatrix} \qquad J_{3,1} = [3]$$

$$J_{-4,4} = \begin{bmatrix} -4 & 1 & & \\ & -4 & 1 & \\ & & -4 & 1 \\ & & & -4 \end{bmatrix}$$

$$A = SJS^{-1}$$

$$J = \begin{bmatrix} \lambda & 1 & & & & & & \\ & \lambda & 1 & & & & & \\ & & \lambda & & & & & \\ & & & \lambda & & & & \\ & & & & \lambda_2 & 1 & & \\ & & & & & \lambda_2 & 1 & \\ & & & & & & \lambda_2 & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_n \end{bmatrix}$$

$$J = \begin{bmatrix} J_{2,1} & & \\ & J_{2,2} & \\ & & J_{3,2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}$$

almost diagonalization except for these 1's.

A 1 in the superdiagonal corresponds to a lack of an eigenvector

But $\lambda = 2$ only lacked 1 eigenvector

$\lambda = 2$ $\lambda = 2$ $\lambda = 2$

$\lambda = 3$ $\lambda = 3$ lacked 1

Let's take $A = \begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix}$

Spoiler ...

$\lambda = 0, \lambda = 0$

$\lambda = 2$

$$A = S J S^{-1}$$

$$= S \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{2} \end{pmatrix} S^{-1}$$

we need one more eigenvector for $\lambda = 0$.

$V_0 = \text{span}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ despite $\lambda = 0$, $\lambda = 0$

added a "kind of eigenvector" which added a 1 above the diagonal.

Normally $A = \begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix}$

$A = SJS^{-1}$, $S = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$

eigenvectors as columns

$\lambda = 0$, $\lambda = 0$ $\qquad$ $\lambda = 2$

$v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

??

$$S = \begin{pmatrix} 1 & & -1 \\ 0 & v_2 & 1 \\ 2 & & 0 \end{pmatrix}$$

??

v_2 is a generalized eigenvector

$A = SJS^{-1}$

$AS = SJ$

$$A = \begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 4 & -3 \\ -4 & -2 & 2 \\ 4 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & & -1 \\ 0 & v_2 & 1 \\ 2 & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & -1 \\ 0 & v_2 & 1 \\ 2 & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 2 \end{pmatrix}$$

$$\left(A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \; Av_2 \; A\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 & 1 & -2 \\ & 0 & 2 \\ & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \vec{0} & Av_2 & \begin{matrix} -2 \\ 2 \\ 0 \end{matrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ & 0 & 2 \\ & 2 & 0 \end{pmatrix}$$

$$\underline{Av_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} \qquad v_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

Given a linear transformation

$T : V \longrightarrow W$, vectorspaces

$\langle v, \tilde{v} \rangle_1$ $\langle w, \tilde{w} \rangle_2$

The adjoint $T^* : W \longrightarrow V$ is

such that $\langle T(v), w \rangle_2 = \langle v, T^*(w) \rangle_1$

Formula if $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$x^T K x$ $y^T L y$

K, L pos. def.

$$A^* = K^{-1} A^T L$$

$n \times m$ $n \times n$ $n \times m$ $m \times m$

In 7.5.1 $K = L$

If dot product, then $A^* = A^T$

$$R_\theta = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

$$\cos\theta^2 - 2\cos\theta\,\lambda + \lambda^2 + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

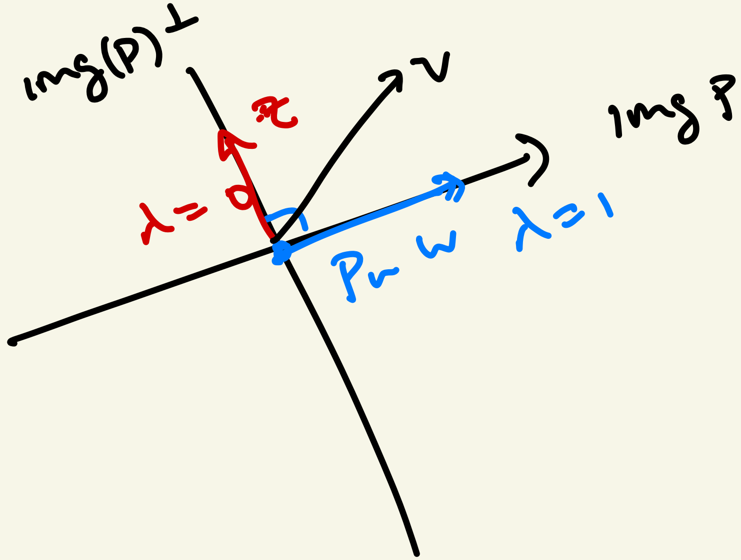
$$\lambda = \frac{\pm\sqrt{}}{}$$

$$\operatorname{img}(P)^\perp = \operatorname{coker}(P)$$

$v \in \operatorname{img}(P)^\perp \Rightarrow Pv = 0.$

If Pv, $v \in \operatorname{img}(P)^\perp$.

then $Pv \in \operatorname{img}(v)$



Let $v = w + z$ $\quad w \in \operatorname{img} P$

$z \in \operatorname{img} P^\perp$

$w - w' \cdot P$

$Pw' - w' \in \ker P$

$$Pv = Pw + Pz$$
$$= PPw' + Pz$$
$$Pv = Pw' + Pz$$

$v - w' - z \in \ker P$

$w + z - w' - z \in \ker P$

$w - w' \in \ker P$

$$P_v \cdot (v - P_v) = P_v \cdot v - P_v \cdot P_v$$

$$= v^T P_v^T - v^T P^T P v = 0$$

$$v^T (P^T - P^T P) v = 0$$

$$A^+ = K^{-1} A K$$

$(v_1, v_2) = e_i$

$2 v_1 w_1 + 3 v_2 w_2$

$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = e_j$

$$= (v_1, v_2) K \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$e_i^T K e_j = k_{ij}$

$\shortparallel$

$$K = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} w_1 \\ w_2 \end{array} & \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \end{array}$$

$$K = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\underline{Aw} = A\,cv = cAv$$

$$= c\lambda v$$

$$= \lambda(cv)$$

$$= \underline{\lambda w}$$

$w = cv$ is an eigenvector

even though $c \in \mathbb{C}$.

$$\det U - \lambda I = \det\begin{pmatrix} u_{11}-\lambda & * & * & * \\ & u_{22}-\lambda & & * \\ & & \ddots & * \\ & & & u_{nn}-\lambda \end{pmatrix}$$

$$= (u_{11}-\lambda)(u_{22}-\lambda)\cdots(u_{nn}-\lambda)$$

Ux 8.2.20

$$A^2 \longrightarrow \lambda^2$$

$$P^2 - \lambda^2 I = P - \lambda^2 I \neq 0$$

V_{λ^2} V_λ compare these.

$$\boxed{\lambda^2 v} = P^2 v = Pv = \boxed{\lambda v} \qquad v \neq 0$$

$$\lambda^2 = \lambda$$

$$\lambda^2 - \lambda = \lambda(\lambda - 1)$$

$$\Rightarrow \lambda = 0, 1$$