Recall : Alternate Gram-Schmidt

$$\{w_1, \dots, w_n\} \longrightarrow \{u_1, \dots, u_n\}$$

orthonormal basis

Recursively solve

$$w_1 = r_{11} u_1$$

$$w_2 = r_{12} u_1 + r_{22} u_2$$

$$\vdots$$

$$w_n = r_{1n} u_1 + \dots + r_{nn} u_n$$

$$r_{11} = \|w_1\| \implies u_1 = \frac{w_1}{\|w_1\|}$$

$$r_{12} = \langle w_2, u_1 \rangle \qquad r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2}$$

$$u_2 = \frac{w_2 - r_{12} u_1}{r_{22}}$$

Step for finding u_j

$$r_{ij} = \langle w_j, u_i \rangle \quad (i < j) \quad *$$

$$r_{jj} = \sqrt{\|w_j\|^2 - \sum_{i=1}^{j-1} r_{ij}^2}$$

$$u_j = \frac{w_j - r_{1j} u_1 - \cdots - r_{j-1,j} u_{j-1}}{r_{jj}}$$

---

let A be a nonsingular matrix

$A = (a_1 \cdots a_n)$, then

(columns of A form a basis of $\mathbb{R}^n$)

$$\begin{cases} a_1 = r_{11} u_1 \\ \vdots \\ a_n = r_{1n} u_1 + \cdots + r_{nn} u_n \end{cases}$$

where $\{u_1 \cdots u_n\}$ is an orthonormal basis.

Let $Q = (u_1 \cdots u_n)$, matrix

w/ columns given by the orthonormal basis the Gram-Schmidt gives us

from $\{a_1, \ldots, a_n\}$.

Note that Q is an orthogonal.

$A = Q\underbrace{R}$! Goal

Recall that:

$$Mc = c_1 m_1 + c_2 m_2 + \cdots + c_n m_n$$

where $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

and m_i is the i^{th} column of M.

$$a_1 = r_{11} u_1 \rightsquigarrow a_1 = (u_1 \cdots u_n)\begin{pmatrix} r_{11} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ast$$

$$(u_1 \cdots u_n)\begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = r_{11}u_1 + 0u_2 + \cdots + 0u_n$$

$$= a_1. = Q\begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So $a_1 = Q\begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

$a_2 = r_{12}u_1 + r_{22}u_2$

$$\rightsquigarrow \quad a_2 = (u_1 \dots u_n)\begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Because

$$Q\begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = r_{12}u_1 + r_{22}u_2 + 0u_3 + \dots + 0u_n$$

$$= r_{12}u_1 + r_{22}u_2 = a_2.$$

$$a_2 = Q\begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In general

$$a_j = Q\begin{pmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{jj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $a_j = r_{1j}u_1 + \dots + r_{jj}u_j$

Remember in general,

$$A(b_1 \dots b_n) = (Ab_1 \dots Ab_n)$$

Applying this formula here!

$$A = (a_1 \dots a_n)$$

$$= \left(Q\begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots\dots Q\begin{pmatrix} r_{1n} \\ r_{2n} \\ \vdots \\ r_{nn} \end{pmatrix} \right)$$

$$= Q\begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & & r_{2n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & & r_{n} \end{pmatrix}$$

$$A = QR = (u_1 \dots u_n)\begin{pmatrix} r_{11} & & r_{1n} \\ & \ddots & \\ 0 & & r_{nn} \end{pmatrix}$$

So we've shown that any nonsingular matrix can be decomposed as $A = QR$. Q is orthogonal, R is upper Δ.

Thm Let A be a nonsingular matrix. Then $A = QR$ where Q is orthogonal and R is upper Δ. This is unique if the diagonal entries of R are positive.

Pf $Q = (u_1 \ldots u_n)$

$$R = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{pmatrix} \checkmark$$

Uniqueness

Given $A = (a_1 \ldots a_n)$, the G-S algorithm applied to $\{a_1 \ldots a_n\}$ will spit out Q and R, and they're almost unique since they were determined by A.

$$r_{11} = \|w_1\| \qquad \boxed{u_1 = \frac{w_1}{\|w_1\|}}$$

$$r_{ij} = \langle w_j, u_i \rangle$$

$$r_{jj} = \sqrt{\|w_j\|^2 - \sum r_{ij}^2}$$

$$u_j = \frac{w_j - \sum r_{ij} u_j}{r_{jj}}$$

The only nonuniqueness of the recursive solving method is when making it a unit vector.

$$u_1 = \frac{w_1}{\|w_1\|} \quad \text{OR} \quad u_1 = -\frac{w_1}{\|w_1\|}$$

$$r_{11} = \underbrace{\|w_1\|} \quad \text{OR} \quad r_{11} = -\|w_1\|$$

$$r_{jj} = \pm\sqrt{\|w_j\|^2 - \sum r_{ij}^2}$$

If we pick the + root, then we get a unique QR. $\square$

Ex

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

$$r_{11} = \sqrt{3} \qquad r_{12} = \frac{1}{\sqrt{3}} \qquad r_{13} = \frac{1}{\sqrt{3}}$$

$$r_{22} = \frac{2\sqrt{2}}{\sqrt{3}} \qquad r_{23} = \frac{-5}{\sqrt{6}}$$

$$r_{33} = \frac{1}{\sqrt{2}}$$

$$u_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow$$

$$A = QR$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \end{pmatrix} R,$$

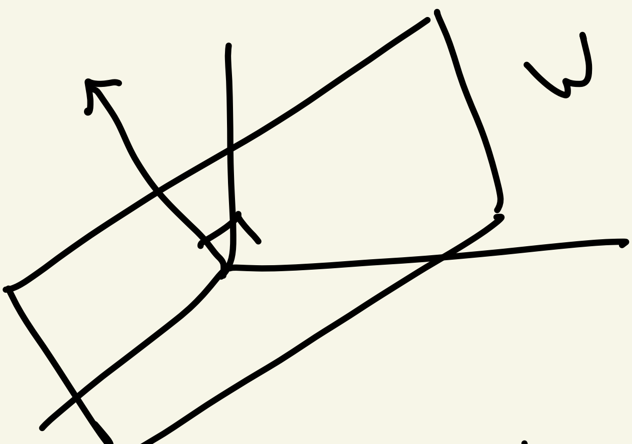
$$R = \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2\sqrt{2}}{\sqrt{3}} & \frac{-5}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

§4.4 Orthogonal Projection

<u>Def</u> Let $W \subseteq V$ be a subspace of an inner product space V.

We say a vector z is orthogonal to W if $\langle z, w \rangle = 0$ $\forall w \in W$.

($\forall$ = for all)



The normal vector to plane is orthogonal to that plane.

z is orthogonal to W if

$\langle z, w_i \rangle = 0 \quad \forall$ basis vectors w_i.

Let $\{w_1 \ldots w_k\}$ be a basis of W.

Then if $\langle z, w_i \rangle = 0 \underbrace{\forall w_i}_{\text{finite}}$

then $\langle z, w \rangle = 0, \underbrace{\forall w \in W}_{\infty \text{ amount of vectors}}$

Pf $\langle z, w \rangle = \langle z, c_1 w_1 + \ldots + c_k w_k \rangle$

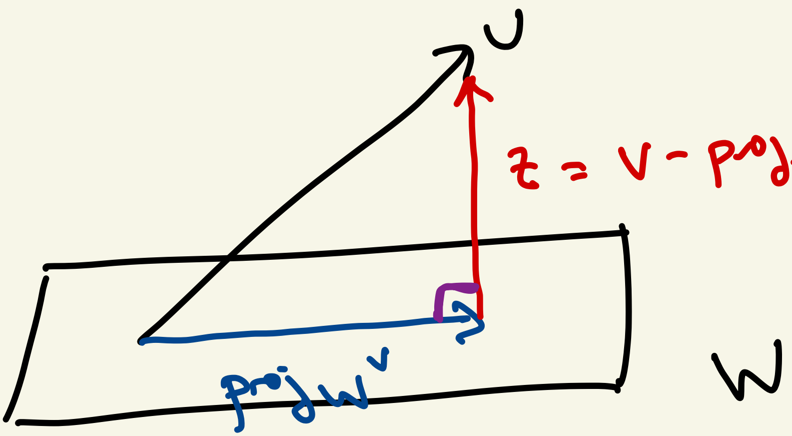
$= c_1 \langle z, w_1 \rangle + \ldots + c_k \langle z, w_k \rangle$

$= 0$ $\square$

<u>Def / Thm</u>: Let $W \subseteq V$ be a f.d subspace in an inner product space V. Then the orthogonal projection of a vector $v \in V$ is

$\text{proj}_W v \in W$ s.t.

$z = v - \text{proj}_W v$ is orthogonal to W.



This is well-defined!

Pf $\text{proj}_W v$

$= c_1 u_1 + \dots + c_k u_k$

where $u_1 \dots u_k$ is an orthonormal basis of W and

$$c_i = \langle v, u_i \rangle$$

More generally, if $\{u_1 \dots u_k\}$ is just orthogonal, then

$$c_i = \frac{\langle v, u_i \rangle}{\|u_i\|^2}.$$

W is f.d, $\{w_1 \dots w_k\} \xrightarrow[G-S]{} \{u_1 \dots u_k\}$ orthog.

So the formula

$$\text{proj}_W v = c_1 u_1 + \dots + c_k u_k$$

make sense.

Let $\{u_1, \dots, u_k\}$ be orthogonal, be a basis of W

Then

$v - \text{proj}_W v \perp W$, supposedly.

For ~~any~~ $w \in W$, for any u_i

$$\langle v - \text{proj}_W v, u_i \rangle$$

$$= \left\langle v - \frac{\langle v, u_1 \rangle}{\|u_1\|^2} u_1 - \dots - \frac{\langle v, u_k \rangle}{\|u_k\|^2} u_k, u_i \right\rangle$$

$$= \left\langle v - \sum_{i=1}^{k} \frac{\langle v, u_i \rangle}{\|u_i\|^2} u_i, u_j \right\rangle$$

0 if $i \neq j$

$$= \langle v, u_j \rangle - \sum_{i=1}^{k} \frac{\langle v, u_i \rangle}{\|u_i\|^2} \cancel{\langle u_i, u_j \rangle}$$

$$= \langle v, u_j \rangle - \frac{\langle v, u_j \rangle}{\cancel{\|u_j\|^2}} \cancel{\langle u_j, u_j \rangle}$$

$$= \langle v, u_j \rangle - \langle v, u_j \rangle = 0$$

So $v - \mathrm{proj}_W v \perp W$ as desired.

We'll see the uniqueness proof a little later.

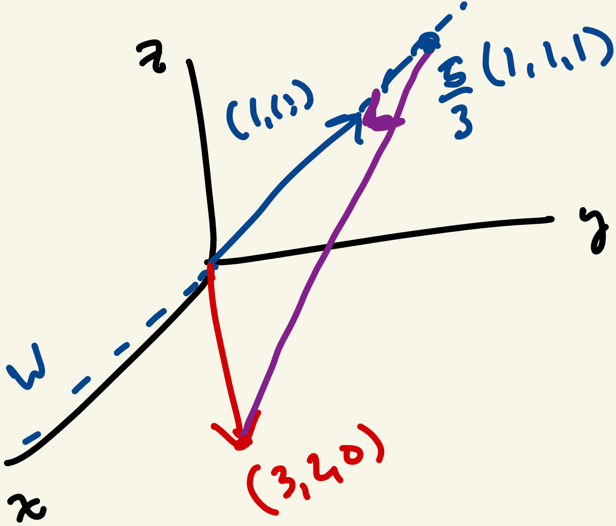
If you picked two different bases, $\{u_1, \dots u_k\}$, $\{u_1', \dots \tilde{u}_k\}$, we get the same $\mathrm{proj}_W v$.

Let $V = \mathbb{R}^3$ $W = \mathrm{span}(1,1,1)$ $v = (3,2,0)$

then

$$\mathrm{proj}_W v = \frac{(3,2,0)\cdot(1,1,1)}{\|(1,1,1)\|^2}(1,1,1)$$

$$= \frac{5}{3}(1,1,1).$$



$v - \text{proj}_W v \perp W$

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \frac{5}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$z = v - \text{proj}_W v$ is orthog to W.

Orthogonal Subspaces

Let $W \subset V$, V inner product space.

Let Z be another subspace of V.

We say Z is orthogonal to W

if $\forall z \in Z$, $\forall w \in W$,

$$\langle z, w \rangle = 0.$$

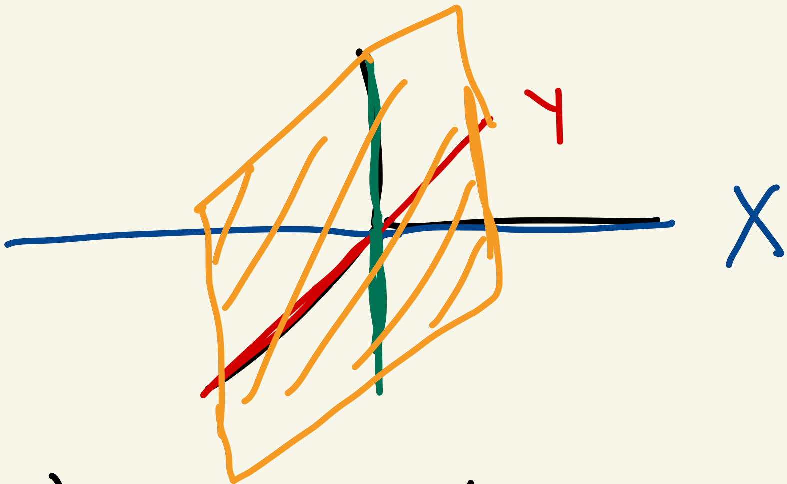
Ex $\operatorname{Span}(4, 1, -5) \perp \operatorname{span}(1, 1, 1)$

Ex $\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \perp \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

since $\begin{pmatrix} a & b & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix} = 0.$

Given a subspace W, there's a "biggest" space orthogonal to it.

$\mathbb{R}^3$



$X \perp Y$ and

$X \perp Z$.

$X \perp$ to the Y-Z plane

$= \text{Span}\{(0,1,0), (0,0,1)\}$

and that's everything $\perp$ to X.

Def / Prop

Let W be a subspace of V, an inner product space.

Define $W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$

= everything orthog to W.

$W^\perp$ is another subspace.

Pf $W^\perp \neq \phi$, $0 \in W^\perp$ ①

$\langle 0, w \rangle = 0 \quad \forall w \in W$.

② Let $v, u \in W^\perp$. We show that $v + u \in W^\perp$.

$v \in W^\perp \quad u \in W^\perp$

$\langle v + u, w \rangle = \cancel{\langle v, w \rangle} + \cancel{\langle u, w \rangle}$

$= 0 + 0 = 0. \quad \forall w.$

$v + u \in W^\perp$.

③ let $c \in \mathbb{R}$, $v \in W^{\perp}$

$$\langle cv, w\rangle = c\langle v, w\rangle = c \cdot 0 = 0.$$

Thus $cv \in W^{\perp}$. $\square$

<u>Ex</u> let $W = \operatorname{span}(1,0,0) \subseteq \mathbb{R}^3$

Claim: $W^{\perp} = y\text{-}z$ plane

$$= \operatorname{span}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right).$$

$$W^{\perp} = \{ v \in \mathbb{R}^3 \mid v \cdot (1,0,0) = 0 \}.$$

$$\left\{ \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \right\}$$

$$= \ker\left(\begin{pmatrix}\underline{1} & \bar{0} & \bar{0}\end{pmatrix}\right)$$

free variables, b, c free.

$\Rightarrow$ $a = 0$, b, c free

$$\Rightarrow \begin{pmatrix}0\\b\\c\end{pmatrix} = b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$W^\perp$ is usually called "W perp".

Prop $W \cap W^\perp = 0$.

Pf Let $w \in W \cap W^\perp$.

Then since $w \in W^\perp$, then

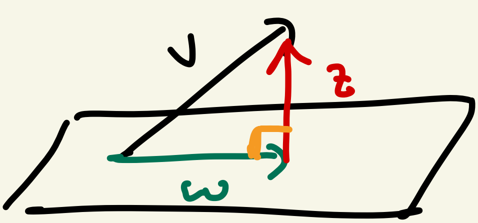
$$\langle w, w \rangle = 0 \implies w = 0. \qquad \square$$

Prop Suppose $W \subseteq V$. Then $\forall v$

v can be uniquely factored

into $v = w + z$,

$w \in W$, $z = W^\perp$.



Pf. let $v = w + z$,
$= \tilde{w} + \tilde{z}$

$w, \tilde{w} \in W$
$z, \tilde{z} \in W^{\perp}$.

$w + z = \tilde{w} + \tilde{z}$
$W \ni w - \tilde{w} = \tilde{z} - z. \in W^{\perp}$.

$w - \tilde{w} \in W$. $\tilde{z} - z \in W^{\perp}$.

$w - \tilde{w} \in W \cap W^{\perp}$
$\tilde{z} - z \in W \cap W^{\perp}$ since they're equal.

Since $W^{\perp} \cap W = 0$
$w - \tilde{w} = 0$
$\tilde{z} - z = 0$
$\Longrightarrow$ $w = \tilde{w}$
$z = \tilde{z}$.

Uniqueness.

To show that $v = w + z$ is

first place, let

$$w = \mathrm{proj}_W v \in W.$$

$$z = v - \mathrm{proj}_W v \in W^\perp \text{ by definition.}$$

So this factorization exists and

it's unique.

($\mathrm{Proj}_W v$ is unique!)

<u>Ex</u> $W = \mathrm{span}\,(1,1,1)$

then $\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \underset{\in W}{\frac{5}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} + \underset{\in W^\perp}{\frac{1}{3}\begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}}.$

<u>Prop</u> If $\dim V = n$, $\dim W = m$

then $\dim W^{\perp} = n - m$.

So it makes sense to call W and $W^{\perp}$ <u>complementary</u> subspaces.

<u>Pf</u> Since $W \cap W^{\perp} = 0$

$$\Longrightarrow \dim(W + W^{\perp}) \quad *$$

$$= \dim(W) + \dim(W^{\perp}).$$

But $\forall v \in V$ $v = w + z$, $w \in W$, $z \in W^{\perp}$.

So $W + W^{\perp} = V$.

$\dim V = \dim W + \dim W^{\perp}$

$\dim W^{\perp} = n - m$. $\square$

Prop If W is finite dimensional, then $(W^\perp)^\perp = W$. V is f.d also.

Pf By definition, given $w \in W$,

then $\langle w, z \rangle = 0 \ \forall z \in W^\perp$

$\Rightarrow w \in (W^\perp)^\perp$.

$$(W^\perp)^\perp = \{ v \in V \mid \langle v, z \rangle = 0 \ \forall z \in W^\perp \}$$

$$W \subseteq (W^\perp)^\perp.$$

Now we need $(W^\perp)^\perp \subseteq W$.

Let $w \in (W^\perp)^\perp$. Since W is f.d, we can project onto it.

Let $w = \underline{proj_W w + z}$, $z \in W^\perp$.

want $z = 0$.

Given a basis $\underline{u_1 \dots u_k}$ of $\underline{W^\perp}$ is orthonormal.

$$w = \mathrm{proj}_W w + z \qquad z \in W^\perp.$$

$$z = \mathrm{proj}_{W^\perp} w.$$

$$z = \langle w, \cancel{u_1} \rangle u_1 + \dots + \langle w, \cancel{u_k} \rangle u_k$$

$$= 0.$$

$$\Rightarrow w = \mathrm{proj}_W w$$

$$\Rightarrow w \in W.$$

$$(W^\perp)^\perp \subseteq W.$$

$$W = (W^\perp)^\perp.$$

$\square$

Non-Example

let $V = C^0[a,b]$ (inf dimensional)

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

let $W = P^{(\infty)}$ = all polynomial functions on $[a,b]$.

$$W^\perp = \left(P^{(\infty)}\right)^\perp$$

$$= \left\{ f \;\middle|\; \int_a^b f(x) p(x)\, dx = 0 \quad \forall p \in P^{(\infty)} \right\}$$

Claim: $W^\perp = 0$.

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \frac{1}{3!} f'''(0) x^3 + \cdots .$$

Despite f being only cts.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Let $P_n(x) = a_0 + a_1 x^1 + \cdots + a_n x^n$.

$P_0 = a_0$

$P_1 = a_0 + a_1 x$

$P_2 = a_0 + a_1 x + a_2 x^2$

$\vdots$ etc

If $f \in W^\perp$ $\displaystyle\int_a^b f(x) P_n(x)\, dx = 0.$

But $P_n(x) \longrightarrow f$ as $n \to \infty$

$$\lim_{n\to\infty} \int_a^b f(x) P_n(x)\, dx = \int_a^b f(x)^2\, dx$$
$$= \| f(x) \|^2$$

So on the one hand

$$\lim_{n\to\infty} \langle f, p_n \rangle$$

$$= \lim_{n\to\infty} \int_a^b f(x) p_n(x)\, dx$$

$$= \int_a^b f(x)^2\, dx = \|f\|^2$$

$$\lim_{n\to\infty} \langle f, p_n \rangle = \lim_{n\to\infty} 0 = 0.$$

since $f \perp p_n$.

$$\|f\|^2 = 0. \implies f = 0.$$

$$(P^{(\infty)})^{\perp} = 0.$$

$$((P^{(\infty)})^{\perp})^{\perp} = 0^{\perp} = C^0[a,b]$$

$$\neq P^{(\infty)}.$$

In infinite dim vector spaces

$(W^\perp)^\perp \neq W^\perp$.

only $W \subseteq (W^\perp)^\perp$.

polynomials

$\subset ((\text{polynomials})^\perp)^\perp$

in $C^0[a,b]$

since $(P^{(\infty)})^\perp = 0$.

<u>Recall</u> Given an $m \times n$ matrix A, m rows, n column

$$\ker(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

$$\operatorname{img}(A) = \{ Ax \in \mathbb{R}^m \}.$$

$$\operatorname{Img}(A) = \operatorname{span}(\text{columns of } A)$$

You could find a basis of $\operatorname{Img}(A)$ by computing the independent columns.

What if we did rows instead of columns?

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \vec{b}.$$

$$Ax = b \xrightarrow{\text{Transpose}}$$

$$x^T A^T = b^T.$$

$$(x \; y \; \cdots \; z) \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} = (\cdots)$$

Each linear equation corresponds to a column of A^T

row reduction $\rightsquigarrow$ column reduction

What if we had a matrix A.
How do $Ax = b$ and $x^T A = b^T$ compare?

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \end{pmatrix}$$ would be.....

$$x + 2y + 3z = 0$$
$$5x + 4y + 6z = 0$$

OR

$$x + 5y = 0$$
$$2x + 4y = 0$$
$$3x + 6y = 0$$

How do the solutions to these compare? *

<u>Corollary</u> <u>to a Thm</u>

$$\dim(\text{span}(\text{rows of } A))$$
$$= \dim(\text{span}(\text{columns of } A))$$

dim(span(rows of A))

$=$ dim(span(columns of A^T))

$\Rightarrow$ rank(A) = dim(img(A))

= dim(span(columns))

Corollary

$$\text{rank}(A) = \text{rank}(A^T)$$

A $m \times n$

A^T $n \times m$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \end{pmatrix}$$

only 2 independent columns

$r_1 + r_2 = r_3$

rank$(A^T) = 2$ since 2 independent rows.

Finish 4 tomorrow...