Final Exam

7/31, 10:10am – $\boxed{12:10\text{pm}}$
+ time to upload

If this is a scheduling conflict
email me!!

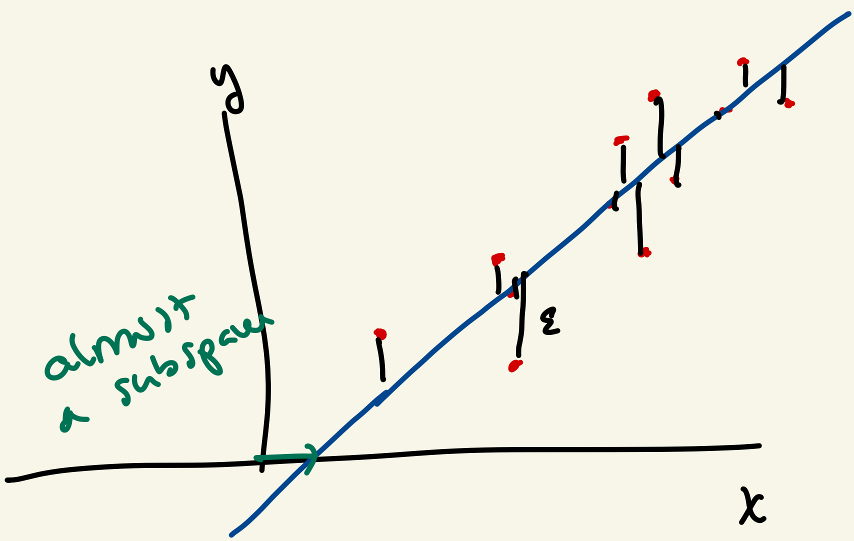
Topics

- Majority past Exam 2 problems
- few problems from Exams 1 and 2.

7/8/9 problems

6 after exam 2   2/3 from Exams 1, 2

Use linear algebra to derive a least squares formula



ε = error

linear regression,

linear approximation

Idea: minimize $\underline{\sum \varepsilon_i^2}$

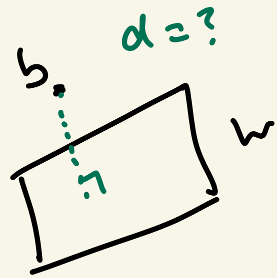
$\Rightarrow$ minimization in linear algebra?

how to minimize a quadratic eq?

how to minimize a quadratic eq?

$\Updownarrow$

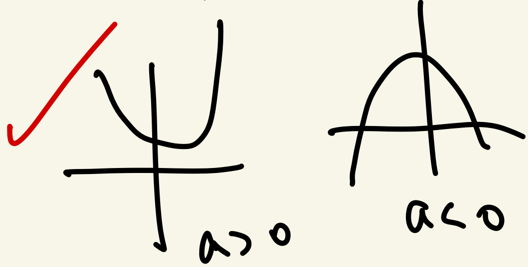
how to minimize the distance between a given subspace W and sample vector b?



5.2 Minimization of quadratics in more than 1 variable.

$$p(x) = ax^2 + bx + c$$

$p'(x) = 0 \Rightarrow$ if $\boxed{a>0}$ then you get a min!



let's say

$$p(x,y) = 2x^2 + 2xy + 2y^2 + 3x - 5y + 2$$

Minimum value?

Can we get this to look like $ax^2 + bx + c$

$q(x) = x^T K x$, K symmetric

$$ax^2 + bx + c$$

$$= x^T K x + x^T b + c$$

$-2 \downarrow -\frac{1}{2} b$

$$= x^T K x - 2x^T f + c$$

(easier to minimize)

$$p(x,y) = \underbrace{2x^2 + 2xy + 2y^2}_{} + \underbrace{3x - 5y}_{} + 2$$

$$\begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}_{}$$

$$\begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$\underbrace{-2 \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}}_{} + 2$$

$ax^2 + bx + c$ has a min $\iff$ $a > 0$

$p(x) = x^T K x - 2x^T f + c$ has a min $\iff$ K is positive definite.

In order for $p(x,y)$ to have a minimum, we need $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ to be positive definite.

Thm let $p(x) = x^T K x - 2x^T f + c$ *

$x \in \mathbb{R}^n$, K is $n \times n$ symmetric

$f, c \in \mathbb{R}^n$ given.

If K is pos. def. then

$p(x)$ has a unique minimum value achieved at $\boxed{x^* = K^{-1} f.}$

(Positive def. matrices are always invertible!)

$$p(x^*) = p(K^{-1}f) = c - f^T x^* .$$

Pf: By definition $f = Kx^*$.

$$p(x) = x^T K x - 2x^T \overset{f}{\overset{\shortparallel}{(Kx^*)}} + c$$

$$= x^T K x - 2x^T K x^* + c$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$p(x) = x^T K x - 2x^T K x^* + c$$

$$= (x - x^*)^T K (x - x^*) + (c - x^{*T} K x^*)$$

$$= (x^T K x - 2x^T K x^* + x^{*T} K x^*) + c - x^{*T} K x^*$$

$$= ((x - x^*)^T K (x - x^*)) + (c - x^{*T} K x^*)$$

$$= \underbrace{(x - x^*)^T K (x - x^*)}_{\text{just need to minimize this part}} + \underbrace{(c - x^{*T} K x^*)}_{\text{constant}}$$

since K is positive definite $\Rightarrow$

$$q(x) = (x - x^*)^T K (x - x^*) > 0$$

$$\forall \; x - x^* \neq 0$$

and when $x = x^* \Rightarrow q(x) = 0$

$q(x)$ has a minimum at x^*!

$\Rightarrow$ $p(x)$ has minimum at $x = x^*$ as well.

$$p(x^*) = p(K^{-1} f)$$

$$= c - \underbrace{x^{*T} K}_{} x^*$$

$$= c - f^T x^* . \qquad \square$$

Ex $p(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$- 2 \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix} + 2$$

$p(x,y)$ has minimum at

$$x^* = K^{-1} f$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -11 \\ 13 \end{pmatrix}$$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is positive definite!

(check eigenvalues > 0)

$$p(K^{-1}f) = c - f^T x^*$$

$$= 2 - \frac{1}{6} \begin{pmatrix} -\frac{3}{2} & \frac{5}{2} \end{pmatrix} \begin{pmatrix} -11 \\ 13 \end{pmatrix}$$

$$= -\frac{2}{3}$$

5.3 Minimizing distance from a point to a subspace

Problem Given a subspace W of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, minimize

$$\|w - b\| \quad \text{over all } w \in W.$$

It's the same as minimizing

$$\|w - b\|^2.$$

Furthermore let W have a basis

$$w_1 \dots w_k.$$

$$w = x_1 w_1 + \dots + x_k w_k.$$

Minimizing over w

$\Longleftrightarrow$ minimizing over $x_1, \dots x_k$.

$p(x) = \|w - b\|^2$ is a function of $x_1 \dots x_k$.

2 ways to write

$p(x) = \|w - b\|^2$ as a quadratic.

since $w = x_1 w_1 + \dots + x_k w_k$

then

$$p(x) = \|w - b\|^2 = \langle w - b, w - b \rangle$$

$$= \|w\|^2 - 2\langle w, b \rangle + \|b\|^2$$

quadratic

$$= \|x_1 w_1 + \dots + x_k w_k\|^2$$

linear

$$- 2\langle x_1 w_1 + \dots + x_k w_k, b \rangle$$

$$+ \|b\|^2$$

$$= \sum_{i,j=1}^{n} x_i x_j \langle w_i, w_j \rangle - 2\sum_{i=1}^{n} x_i \langle w_i, b \rangle$$

$$+ \|b\|^2$$

$$= \underbrace{\sum_{i,j=1}^{n} x_i x_j \langle w_i, w_j \rangle}_{\text{quadratic form assoc. Gram matrix of } w_i} - 2\sum_{i=1}^{n} x_i \langle w_i, b \rangle + \|b\|^2$$

$\|w-b\|^2$

$$= x^T K x - 2 x^T f + \|b\|^2$$

where K = gram matrix of the w_i.

$$K_{ij} = \langle w_i, w_j \rangle$$

why is K positive definite?

$$f = \begin{pmatrix} \langle w_1, b \rangle \\ \langle w_2, b \rangle \\ \vdots \\ \langle w_n, b \rangle \end{pmatrix}$$

minimal value of $\|w-b\|^2$

is $x = K^{-1} f$ if $A = (w_1 \dots w_n)$

$$x = (A^T A)^{-1} f$$

$\sqrt{p(x^*)}$ = minimal distance from W to b is $\qquad x^* = K^{-1}f$

$$= \sqrt{p(K^{-1}f)}$$

$$= \sqrt{\|b\|^2 - f^T K^{-1} f}$$

$$= \sqrt{\|b\|^2 - f^T x^*}$$

<u>Ex</u>: let $W = \text{span}\begin{pmatrix}1\\2\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\2\end{pmatrix} \subseteq \mathbb{R}^4$

let $b = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$: what is the minimal distance from W to b?

Basis for $W = \begin{pmatrix}1\\2\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\2\end{pmatrix}$.

$K = \begin{pmatrix}6 & -1\\-1 & 5\end{pmatrix}$ = gram matrix for W

$$f = \begin{pmatrix}\langle w_1, b\rangle\\ \langle w_2, b\rangle\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

The minimal value occurs at

$$x^* = K^{-1} f$$

$$= \begin{pmatrix} 6 & -1 \\ -1 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{29} \begin{pmatrix} 5 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{29} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

The point of W which is closest to $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

is

$$\frac{1}{29} 5 \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{29} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

That minimal distance is

$$\sqrt{\|b\|^2 - f^T x^*} = \frac{1}{29} \left(2\sqrt{274} \right)$$