§ 9.1

Algebra

Linear algebra

matrices

polynomials

varieties

algebraic geometry

number theory

Analysis

Calculus

diff eq

multivariable

probability

dynamical systems

Linear iterative system

not an exponent but an index

Given an initial vector $u^{(0)} = a$

$$u^{(k+1)} = T u^{(k)}$$

$u^{(k)} \in \mathbb{R}^n$ $\quad T \in M_{n\times n}(\mathbb{R})$

$(\mathbb{C}^n)$ $\quad (M_{n\times n}(\mathbb{C}))$

- $u^{(0)} = a$
- $u^{(1)} = Tu^{(0)} = Ta$
- $u^{(2)} = Tu^{(1)} = T(Ta) = T^2 a$

$\vdots$

- $u^{(k)} = T^k a$

$a, Ta, T^2 a, T^3 a, \ldots$

What does this sequence look like (how does it behave) as a changes? How about as T changes?

Let $T = \lambda I$.

$$u^{(k)} = T^k u^{(0)} = T^k a$$

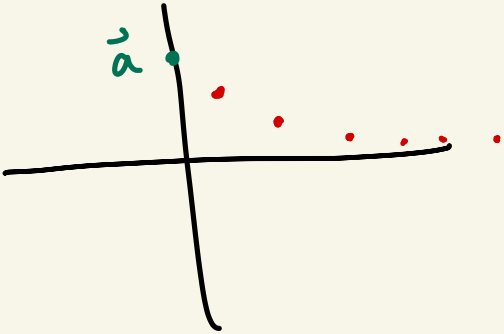
$$= (\lambda I)^k a = \lambda^k a$$

$$a, \ \lambda a, \ \lambda^2 a, \ \lambda^3 a, \ \lambda^4 a, \ldots$$

What happens for different λ values?

$a, \quad \lambda a, \quad \lambda^2 a, \quad \lambda^3 a, \quad \ldots$

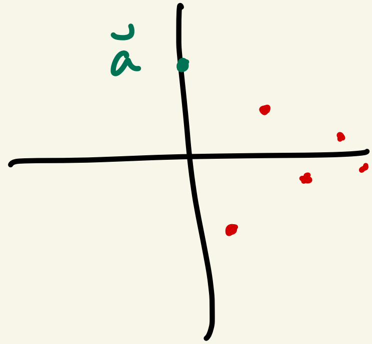
Scalar iterative



$0 < \lambda < 1$

$\lambda^k \longrightarrow 0$

asymptotically stable
stable

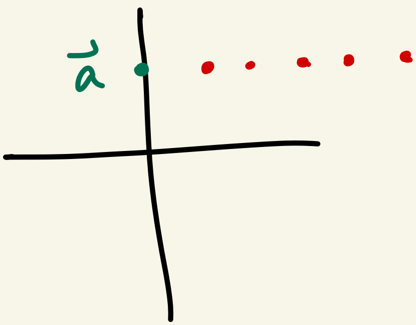


$-1 < \lambda < 0$

$\lambda^k \longrightarrow 0$

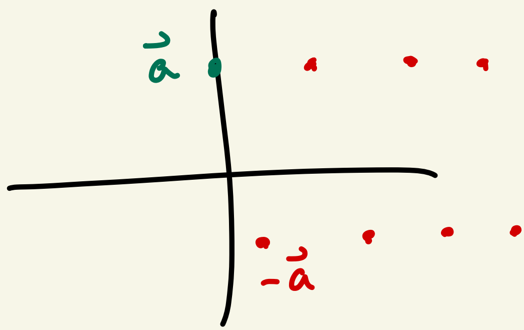
but it alternates

stable, asymptotically stable



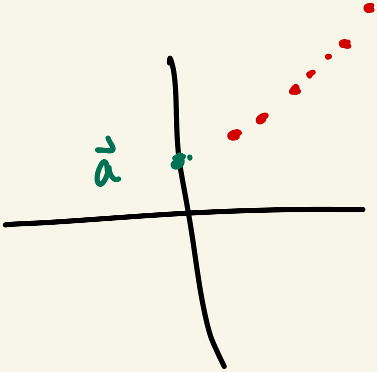
$\lambda = 1$

stable



$\lambda = -1$

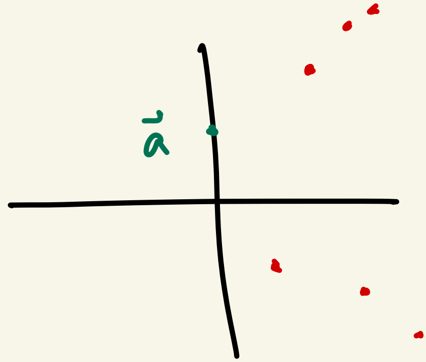
stable



$\lambda > 1$

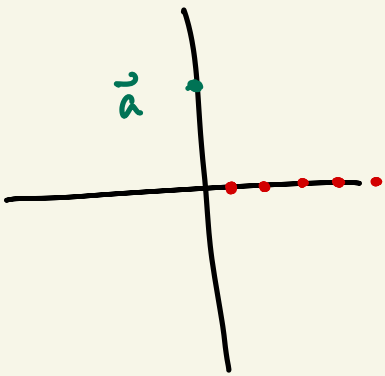
$\lambda^n \to \infty$

unstable



$\lambda < -1$

unstable



$\lambda = 0$

stable

What if T is a more difficult matrix?

Let's say magically $u^{(0)}$ is an eigenvector for T.

$$\underline{u^{(0)}}, \quad u^{(1)}, \quad u^{(2)}, \quad u^{(3)}$$

not usually an eigenvector

$$u^{(1)} = Tu^{(0)} = \lambda u^{(0)}, \quad u^{(2)} = \lambda^2 u^{(0)}$$

$$u^{(0)}, \quad \lambda u^{(0)}, \quad \lambda^2 u^{(0)}, \quad \lambda^3 u^{(0)}$$

Same as a scalar iterative system!

Thm let $u^{(k+1)} = Ju^{(k)}$, $u^{(0)} = a$

be a linear iterative system.

If J is diagonalizable, i.e. has a basis of eigenvectors $v_1, \ldots, v_n$ $(\lambda_1, \ldots, \lambda_n)$

then

$$\left. u^{(k)} = c_1 \lambda_1^k v_1 + \ldots + c_n \lambda_n^k v_n \right\}$$ explicit formula!

where $a = c_1 v_1 + \ldots + c_n v_n$.

c_i are determined by $a = u^{(0)}$.

Pf $u^{(k+1)} = Tu^{(k)}$, $u^{(0)} = a$

$$u^{(k)} = T^k u^{(0)} = \underbrace{T^k a}$$

compute this more explicitly

$\lambda_1 \dots \lambda_n$ w/ a basis of eigenvectors $v_1 \dots v_n$ of T.

Since they form a basis,

$a \in \mathrm{Span}(v_1 \dots v_n)$.

let $a = \underline{c_1} v_1 + \dots + \underline{c_n} v_n$.

determined by $\vec{a}$. can solve using row reduction

Then

$$u^{(1)} = Ta$$

$$= T(c_1 v_1 + \dots + c_n v_n)$$

$$
\begin{aligned}
u^{(1)} &= Ta \\
&= T(c_1 v_1 + \dots + c_n v_n) \\
&= c_1 T v_1 + \dots + c_n \underbrace{T v_n}_{\text{eigenvectors}} \\
&= c_1 \lambda_1 v_1 + \dots + c_n \lambda_n v_n
\end{aligned}
$$

$$
\begin{aligned}
u^{(2)} &= T u^{(1)} \\
&= T(c_1 \lambda_1 v_1 + \dots + c_n \lambda_n v_n) \\
&= c_1 \lambda_1 T v_1 + \dots + c_n \lambda_n T v_n \\
&= c_1 \lambda_1^2 v_1 + \dots + c_n \lambda_n^2 v_n
\end{aligned}
$$

In general

$$
u^{(k)} = c_1 \lambda_1^k v_1 + \dots + c_n \lambda_n^k v_n .
$$

$\square$

So $\lambda_1 \ldots \lambda_n$ determine how $u^{(k)}$ behaves!

Ex

$$u^{(0)} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad T = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}$$

$$u^{(k+1)} = T u^{(k)}$$

$$u^{(k)} = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}^k \begin{pmatrix} a \\ b \end{pmatrix}.$$

What's a formula for $\begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}^k$?

$$\det T - \lambda I = \det \begin{pmatrix} 0.6-\lambda & 0.2 \\ 0.2 & 0.6-\lambda \end{pmatrix} = 0$$

$$\lambda_1 = 0.4 \qquad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 0.8 \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_1 = 0.4 \qquad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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$$u^{(k)} = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}^k \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= c_1 (0.4)^k \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 (0.8)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$c_1 = \frac{b-a}{2} \qquad c_2 = \frac{a+b}{2}$$

$$u^{(k)} = \frac{b-a}{2}(0.4)^k \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{a+b}{2}(0.8)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since $0.4^k \to 0$ $0.8^k \to 0$

w/out alternating

$u^{(k)} \to 0$ w/out alternating

$0.4^k \to 0$ faster than

$0.8^k \to 0$ does.

$\Rightarrow$ For larger k

$$u^{(k)} \sim \frac{a+b}{2}(0.8)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $u^{(k)}$ look more and more like multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note! This method is the same as diagonalizing T.

$$T = S\Lambda S^{-1}. \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$S = (v_1 \cdots v_n)$$

$$u^{(k)} = T^k a$$

$$T^k = (S\Lambda S^{-1})^k$$

$$= \underbrace{(S\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda \cancel{S^{-1}}) \cdots (\cancel{S}\Lambda S^{-1})}_{k \text{ times}}$$

$$= S\Lambda^k S^{-1} = S\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} S^{-1}$$

$$T^k a = S\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \underbrace{S^{-1}a}_{a \text{ in } v_1 \cdots v_n \text{ coordinates}}$$

$$T^k a = S \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n.$$

Same as before.

<u>Ex</u> Explicit formula for the Fibonacci numbers.

$$f_{k+2} = f_{k+1} + f_k \quad \Big| \quad f_0 = 1, \; f_1 = 1.$$

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

As a vector linear recursion

$$f^{(k)} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}$$

$$f^{(k)} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix},$$

$$f^{(k+1)} = \begin{pmatrix} f_{k+1} \\ f_{k+2} \end{pmatrix} = \begin{pmatrix} f_{k+1} \\ f_k + f_{k+1} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}}_{f^{(k)}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} f^{(k)}$$

$$f^{(0)} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad f^{(k)} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}$$

$$f^{(k)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In HW $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

If $f^{(k)} = \begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} \to T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Diagonalize $T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and calculate $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k$ to get a explicit formula for f_k.

Continue tomorrow...