$$f_{k+2} = f_{k+1} + f_k \,, \quad f_0 = 1 \,, \; f_1 = 1$$

$1, 1, 2, 3, 5, 8, 13, \ldots$

$$f^{(k)} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}$$

$$f^{(k+1)} = \begin{pmatrix} f_{k+1} \\ f_{k+2} \end{pmatrix}$$

$$= \begin{pmatrix} f_{k+1} \\ f_{k+1} + f_k \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} f^{(k)}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad f^{(0)} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solve using eigenvalues and vectors!

$$f^{(k)} = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$$

where λ_1 λ_2 are the eigenvalues

v_1 v_2 are the eigenvectors

$\boxed{c_1 c_2}$ where determined by $f^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

But this the same as diagonalization

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \varphi = \text{golden ratio!}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi}$$

$$\frac{1-\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} = \frac{1-5}{2 \cdot 2} = \frac{-4}{4} = -1$$

$$\Rightarrow \frac{1-\sqrt{5}}{2} = \frac{-1}{\frac{1+\sqrt{5}}{2}} = \frac{-1}{\varphi}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$$\lambda = \varphi \qquad v = \begin{pmatrix} 1/\varphi \\ 1 \end{pmatrix}$$

$$\lambda = \frac{-1}{\varphi} \qquad v = \begin{pmatrix} -\varphi \\ 1 \end{pmatrix}$$

$$f^{(k)} = T f^{(k-1)}$$

$$= T^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f^{(k)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k = S \begin{pmatrix} \varphi & \\ & \frac{-1}{\varphi} \end{pmatrix}^k S^{-1}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \frac{-1}{\varphi} \end{pmatrix} \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix}^{-1}$$

$$f^{(k)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$f^{(k)} = \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\frac{1}{\varphi} \end{pmatrix}^k \underbrace{\begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

$$f^{(k)} = \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^k & 0 \\ 0 & \left(\frac{-1}{\varphi}\right)^k \end{pmatrix} \dots$$

$$\begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \frac{1}{\frac{1}{\varphi} + \varphi} \begin{pmatrix} 1 & \varphi \\ -1 & \frac{1}{\varphi} \end{pmatrix}$$

$$= \frac{1}{\frac{-1+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2}} \begin{pmatrix} 1 & \varphi \\ -1 & \frac{1}{\varphi} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \varphi \\ -1 & \frac{1}{\varphi} \end{pmatrix}$$

$$f^{(k)} = \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^k & 0 \\ 0 & \frac{-1}{\varphi^k} \end{pmatrix} \begin{pmatrix} 1 & \varphi \\ -1 & \frac{1}{\varphi} \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1}{\varphi} & -\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^k & 0 \\ 0 & \frac{-1}{\varphi^k} \end{pmatrix} \begin{pmatrix} 1+\varphi \\ -1+\frac{1}{\varphi} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \left(\overbrace{(1+\varphi)}^{c_1} \overbrace{\varphi^k}^{\lambda_1^k} \overbrace{\begin{pmatrix} \frac{1}{\varphi} \\ 1 \end{pmatrix}}^{v_1} + \underbrace{\left(-1+\frac{1}{\varphi}\right)}_{c_2} \underbrace{\left(\frac{-1}{\varphi}\right)^k}_{\lambda_2^k} \underbrace{\begin{pmatrix} -\varphi \\ 1 \end{pmatrix}}_{v_2} \right)$$

$$f^{(k)} = \begin{pmatrix} \boxed{f_k} \\ f_{k+1} \end{pmatrix}$$

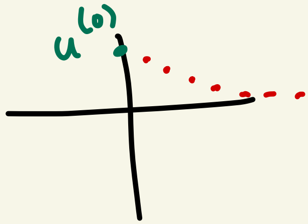
$c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2$

$$f_k = \frac{1}{\sqrt{5}} \left(\varphi^k - \left(\frac{-1}{\varphi}\right)^k \right)$$

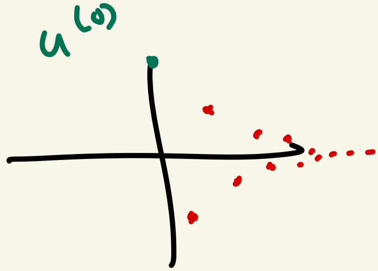
$$\varphi = \frac{1+\sqrt{5}}{2}$$

§ 9.2

If $|\lambda| < 1$ and $u^{(k)} = \lambda u^{(k-1)}$



$1 > \lambda > 0$



$-1 < \lambda < 0$

In general eigenvalues $|\lambda_i| < 0$

make $c_i \lambda_i^k v_i \longrightarrow 0$

and if $|\lambda_i| > 1$ $\quad c_i \lambda_i^k v_i \longrightarrow \infty$

$\lambda_i = \pm 1$ $\quad c_i v_i$

$\pm c_i v_i$

$$u^{(k)} = c_1 \underline{\lambda_1^k} v_1 + \dots + c_n \underline{\lambda_n^k} v_n$$

$u^{(k)} \longrightarrow 0$ in general if

$|\lambda_i| < 1$

Thm The following are equivalent.

Let $u^{(k)}$ be a linear iterative system. $u^{(0)} = \vec{a}$ $u^{(k+1)} = Tu^{(k)}$.

1) $u^{(k)} \longrightarrow 0$ no matter what $\vec{a}$ is

2) $T^k \longrightarrow 0$ as $k \to \infty$

3) All eigenvalues λ_i of T are such that $|\lambda_i| < 1$. ($\lambda \in \mathbb{C}$ possibly)

Note: Outside of the course is what $u^{(k)} \longrightarrow 0$ and $T^k \longrightarrow 0$ means.

$\sum \frac{1}{n}$ diverges

$\sum \frac{1}{2^n}$ converges because it gets close to a number.

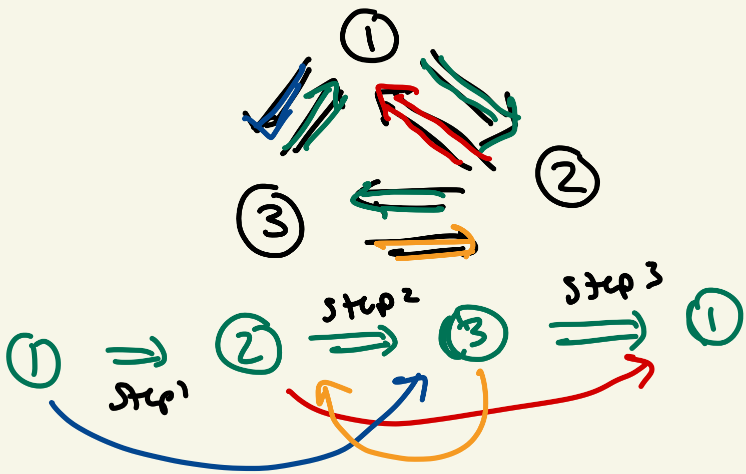
$T^k \longrightarrow 0$ converges to the 0 matrix because it gets "close" to 0 as k gets bigger.

Pf How do you prove "the following are equivalent" type statements?

1) $u^{(k)} \to 0$ as $k \to \infty$

2) $J^k \to 0$

3) $|\lambda_i| < 1$ for all λ_i.



Step 1 ① $\Rightarrow$ ②

Assume $u^{(k)} \to 0 \quad \forall\, u^{(0)} = a$.

In particular if $a = u^{(0)} = e_i$

then $u^{(k)} = J^k e_i$

$u^{(k)} = J^k e_i \longrightarrow 0$ by assumption.

But $J^k e_i$ is the ith column of J^k.

So all columns of $J^k \longrightarrow 0$ individually.

0 vector

This means that $J^k \longrightarrow 0$.

zero matrix

<u>Step 2</u> (2) $\Longrightarrow$ (3)

Assume $J^k \to 0$. In particular

$$J^k v_i \longrightarrow \vec{0}.$$

v_i is the eigenvector for λ_i.

Assume for contradiction that $|\lambda_i| \geq 1$.

Then $J^k = S \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} S^{-1}$

$$J^k v_i = S \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} S^{-1} v_i =$$

$$S \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} e_i = \lambda_i^k v_i$$

By assumption $T^k v_i \to 0$

but if $|\lambda_i| \geq 1$

$$T^k v_i = \lambda_i^k v_i \not\longrightarrow 0$$

contradiction, so $|\lambda_i| < 1$.
$\forall i$.

Step 3 ③ $\Rightarrow$ ①

Assume $|\lambda_i| < 1$. We want to show
$u^{(k)} \to 0$ as $k \to \infty$.

By the formula

$$u^{(k)} = c_1 \lambda_1^k v_1 + \ldots + c_n \lambda_n^k v_n$$

$$\|u^{(k)}\| = \|c_1 \lambda_1^k v_1 + \ldots + c_n \lambda_n^k v_n\|$$

Δ triangle ineq

$$\leq |c_1| \underbrace{|\lambda_1|^k}_{<1} \|v_1\| + \ldots + |c_n| \underbrace{|\lambda_n|^k}_{<1} \|v_n\|$$

$$\longrightarrow 0$$

Since $\|u^{(k)}\| \longrightarrow 0$

therefore $u^{(k)} \longrightarrow \vec{0}$. $\square$

$\lambda = \frac{1}{2}$ $\quad \lambda^k = \left(\frac{1}{2}\right)^k = \frac{1}{2^k} \longrightarrow 0$

$\lambda = 2$ $\quad \lambda^k = 2^k \longrightarrow \infty$

Ex $\quad T = \begin{pmatrix} 1 & -\frac{1}{3} & 2 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & -1 & 6 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

Does $T^k \longrightarrow 0$ as $k \to \infty$?

All we have to do is check that $|\lambda| < 1$ for all the eigenvalues.

$$T = \frac{1}{3}\begin{pmatrix} 3 & -1 & 6 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\lambda_1 = \frac{2}{3} \qquad \lambda_2 = \frac{1}{3} - \frac{1}{3}i \qquad \lambda_3 = \frac{1}{3} + \frac{1}{3}i$$

$$|\lambda_1| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$$

$$|\lambda_2| = \left|\frac{1}{3} - \frac{1}{3}i\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3} < 1$$

$$|\lambda_3| = \left|\frac{1}{3} + \frac{1}{3}i\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

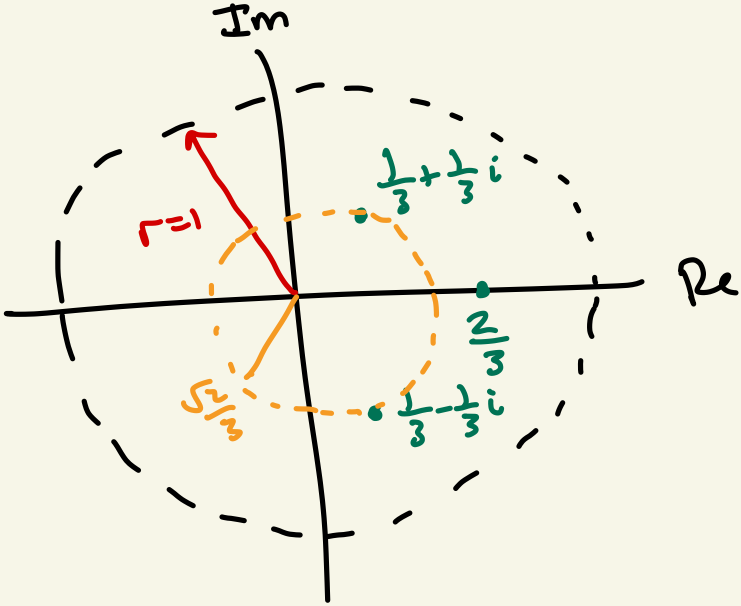
$$= \frac{\sqrt{2}}{3} < 1$$

So all $|\lambda| < 1$, so

$$T^k \longrightarrow 0 \quad \text{as} \quad k \to \infty.$$

$$\underset{= T^k u^{(0)}}{u^{(k)}} \longrightarrow 0$$ no matter what $u^{(0)}$ is.

$\textcircled{\text{I}}$



All the eigenvalues λ need to be in this disc if $J^k \to 0$

$$\lambda = \frac{2}{3} \qquad \lambda = \frac{1}{3} \pm \frac{1}{3}i$$

$\lambda = \frac{2}{3}$ has the biggest absolute value

$$\frac{\sqrt{2}}{3} < \frac{2}{3}.$$

$\left(\frac{2}{3}\right)^k \to 0$ slower than $\left(\frac{1}{3} \pm \frac{1}{3}i\right)^k \to 0$ does.

$$u^{(k)} = c_1 \left(\tfrac{2}{3}\right)^k v_1 + \underbrace{c_2 \left(\tfrac{1}{3} - \tfrac{1}{3} i\right)^k v_2}$$

$(4,-2,1)$

$$+ \underbrace{c_3 \left(\tfrac{1}{3} + \tfrac{1}{3} i\right)^k v_3}$$

As k gets big, these terms matter less and less.

For large k

$$u^{(k)} \approx c_1 \left(\tfrac{2}{3}\right)^k v_1$$

since $\frac{2}{3}$ is biggest.

<u>Def</u>: If $u^{(k)} \longrightarrow 0$ as $k \to \infty$ we say that $u^* = 0$ is <u>globally</u> <u>asymptotically</u> <u>stable</u>.

Def Let T be a matrix w/ eigenvalues $\lambda_1 \dots \lambda_n$.

$$\rho(T) = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

called the <u>spectral</u> <u>radius</u>.

Ex $\rho(T)$, $T = \frac{1}{3}\begin{pmatrix} 3 & -1 & 6 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$$\rho(T) = \max\left\{\left|\tfrac{2}{3}\right|, \left|\tfrac{1}{3} \pm \tfrac{1}{3} i\right|\right\}$$

$$= \frac{2}{3}.$$

In particular $\rho(T) < 1$ and T is <u>diagonalizable</u>, then $T^k \to 0$.

Note: All of today assumes T is diagonalizable.

Fixed points and stability.

let $u^{(k+1)} = Tu^{(k)}$, $u^{(0)} = a$ be a linear iteration system.

We say that u^* is a <u>fixed point</u> iff $Tu^* = u^*$.

Prop u^* is a fixed point iff it's an eigenvector of T w/ eigenvalue $\lambda = 1$.

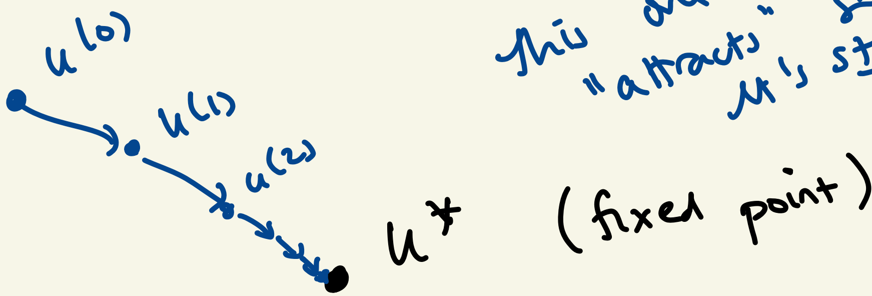
$$\left(Tu^* = u^*\right) \iff \left(Tu^* = \lambda u^*\right)$$
when $\lambda = 1$

So in particular, the set of fixed points $= V_1 =$ set of eigenvectors for $\lambda = 1$

$$= \ker(T - 1I)$$
$$= \ker(T - I).$$

Prop $Tu^{(k)} = u^{(k+1)}$ has a nonzero fixed point iff T has eigenvalue $\lambda = 1$.

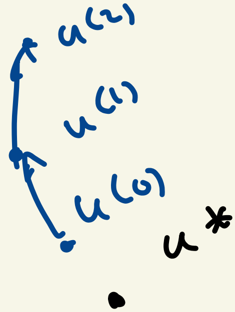
(zero is always a fixed point)



this one "attracts" so it's <u>stable</u>

$u^{(0)} \to u^*$ sometimes.

" $Tu^{(\infty)} = u^{(\infty+1)} = u^{(\infty)}$ "



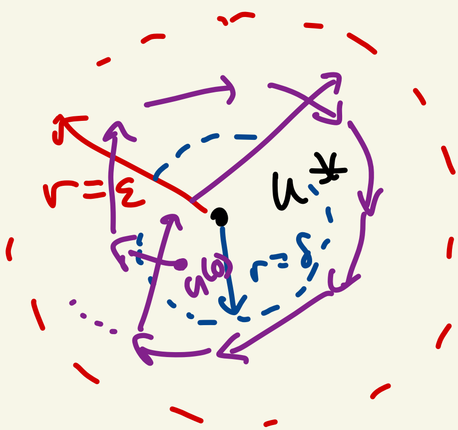
Unstable
not "attractive".

Def: let u^* be a fixed point for T. Then u^* is called **stable** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|u^{(0)} - u^*\| < \delta$

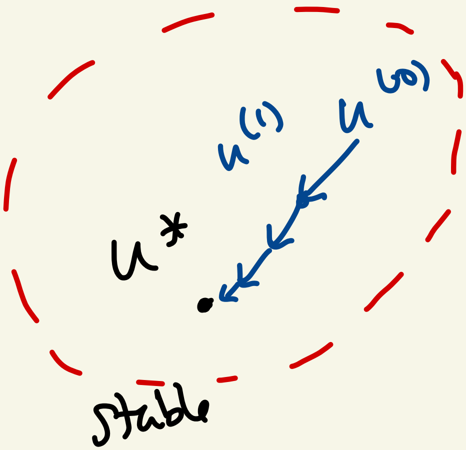
$$\Longrightarrow \quad \|u^{(k)} - u^*\| < \varepsilon \quad \forall k.$$

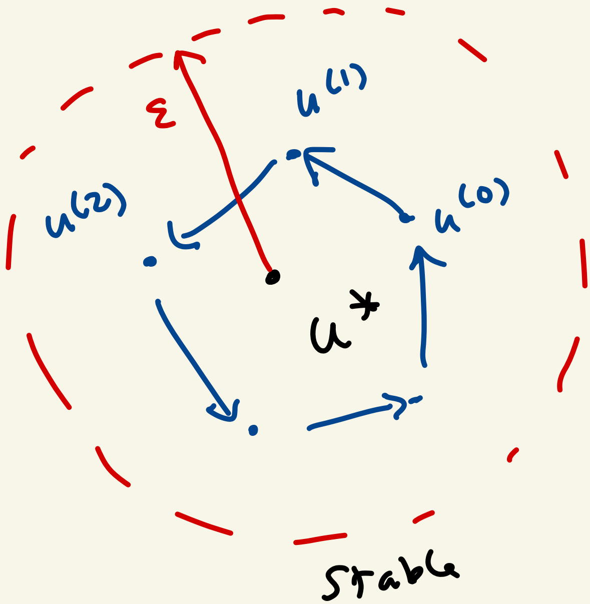
u^* is stable if you want to get all $u^{(k)}$ within ε of u^*, then you can to start within δ.

If I start in δ, the iterative system stays in ε.



Note: If u^* is stable, $u^{(k)}$ need not converge to u^*.





Prop Suppose $\boxed{\rho(T) = 1}$ and $\lambda = 1$ has no repeats. ($\lambda = 1$ is simple)

Then all $u^{(k)} \to u^*$, u^* is a fixed point. Moreover, all fixed points are stable.

Pf Suppose T has eigenvalue $\lambda = 1$

So $V_1 = \ker(T - I) = \operatorname{span}(v_1)$.

Suppose $u^{(0)} = \vec{a}$ and $u^{(k+1)} = Tu^{(k)}$.

Then
$$u^{(k)} = c_1 \underbrace{v_1}_{\lambda_1 = 1} + c_2 \lambda_2^k v_2 + \dots + c_n \lambda_n^k v_n$$

The $|\lambda_2| \dots |\lambda_n| < 1$ so as $k \to \infty$

$$u^{(k)} = c_1 v_1 + \dots + c_n \underbrace{\cancel{\lambda_n^k}}_{0} v_n \longrightarrow c_1 v_1.$$

Since $c_1 v_1 = u^*$ is a fixed point and $u^{(k)} \longrightarrow u^*$.

So why is $u^* = c_1 v_1$ stable?

$\| u^{(k)} - u^* \|$ $\qquad a = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$= \| \cancel{c_1 v_1} + c_2 \lambda_2^k v_2 + \dots + c_n \lambda_n^k v_n - \cancel{c_1 v_1} \|$

$= \| c_2 \lambda_2^k v_2 + \dots + c_n \lambda_n^k v_n \|$

let λ_j be the second biggest

$\leq |\lambda_j|^k \left(|c_2| \|v_2\| + \dots + |c_n| \|v_n\| \right)$

$< \varepsilon$ $\quad$ (make $c_2 \dots c_n$ small enough)

need to be close to $c_1 v_1$

$a \approx c_1 v_1$.

Ex let $T = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -3 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

Find all fixed points of T and show that they're stable.

Need to show $\lambda = 1$ is eigenvalue and $|\lambda_2|, |\lambda_3| < 1$. $(\rho(T) = 1)$

Compute $\lambda_1 \lambda_2 \lambda_3$

$\boxed{\lambda_1 = 1}$ $\quad \lambda_2 = \frac{1}{2} + \frac{1}{2}i \quad \lambda_3 = \frac{1}{2} - \frac{1}{2}i$

$v_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2-i \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2+i \\ -1 \\ 1 \end{pmatrix}$

Fixed points $= \text{Span}\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

u^* are stable!

$|\lambda_2| = |\lambda_3|$

$= \left|\frac{1}{2} + \frac{1}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \frac{\sqrt{2}}{2} < 1$

We know that if $u^{(0)} = c_1 v_1 + \dots + c_n v_n$

then $u^{(k)} \longrightarrow c_1 v_1$.

let $u^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $T = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -3 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

Then $u^{(k)} = T^k u^{(0)} \longrightarrow ???$

Find $\lim_{k \to \infty} u^{(k)}$ in this situation.

We know that $u^{(k)} \longrightarrow u^*$

$u^* = c_1 v_1$

$u^{(k)} \longrightarrow c_1 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. What is c_1?

$u^{(0)} = c_1 v_1 + c_2 v_2 + c_3 v_3$.

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2-i \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2+i \\ -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = c_1\begin{pmatrix}4\\-2\\1\end{pmatrix} + c_2\begin{pmatrix}2-i\\-1\\1\end{pmatrix} + c_3\begin{pmatrix}2+i\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4 & 2-i & 2+i\\-2 & -1 & -1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}4 & 2-i & 2+i\\-2 & -1 & -1\\1 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-2\\ \frac{3}{2}+\frac{3}{2}i\\ \frac{3}{2}-\frac{3}{2}i\end{pmatrix}$$

In particular $c_1 = -2$.

So $J^k\begin{pmatrix}1\\1\\1\end{pmatrix} \longrightarrow -2\begin{pmatrix}4\\-2\\1\end{pmatrix} = \begin{pmatrix}-8\\4\\-2\end{pmatrix}$

where $J = \begin{pmatrix}\frac{3}{2} & -\frac{1}{2} & -3\\ -\frac{1}{2} & \frac{1}{2} & 1\\ \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$.

$$u^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad T = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -3 \\ -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$u^{(1)} = T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$u^{(2)} = T^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = T \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \cdots$$

$$\vdots$$

$$u^{(5)} = \begin{pmatrix} -9.5 \\ 4.75 \\ -2.75 \end{pmatrix}$$

$$u^{(15)} = \begin{pmatrix} -7.9766 \\ 4.0 \\ -2.0 \end{pmatrix}$$

$$u^{(30)} = T^{30} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8.0001\ldots \\ 4.0001\ldots \\ -2.0001\ldots \end{pmatrix} \approx \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix}$$

and

$$T \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \\ -2 \end{pmatrix}.$$

as predicted!

Without computing the eigenvalues, can you look and the matrix J, and see if you can learn anything about J^k or $u^{(k)}$?

Use a matrix norm!

Recall: Given a norm $\|\cdot\|$ on $\mathbb{R}^n$ we can $\|A\|$, $A \in M_{n\times n}(\mathbb{R})$, by the formula

$$\|A\| = \max\left\{ \|Au\| \;\middle|\; u \text{ is a unit vector for } \|\cdot\| \right\}$$

Prop: $\rho(A) \leq \|A\|$.

In particular if $\|A\| < 1$ $\Rightarrow A^k \to 0$ since $|\lambda_i| < 1$.

$(|\lambda_i| \leq \rho(A) \leq \|A\| < 1.)$

Pf: let $\rho(A) = \max\{|\lambda_i|\}$.

If $\lambda \in \mathbb{R}$, pick an eigenvector $u \in V_\lambda$, with $\|u\| = 1$.

$$|\lambda| = |\lambda|\,\|u\| = \|\lambda u\| = \|Au\| \le \max\left\{\|Au\| \;\middle|\; \begin{array}{c}\text{all}\\ \text{unit}\\ \text{vectors}\end{array}\right\} = \|A\|.$$

Pf if $\lambda \in \mathbb{C}$, more annoying. $\square$

Recall L^∞ norm on $\mathbb{R}^n$.

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \dots, |v_n|\}.$$

$$\longrightarrow \|A\|_\infty = \max\{\|Au\|_\infty \mid \|u\|_\infty = 1\}$$

$\|A\|_\infty =$ max absolute row sum.

$$\|A\|_\infty = \max\left\{ \sum_j |a_{ij}| \;\middle|\; i \right\}$$

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{4} \\ -\frac{2}{3} & \frac{1}{5} & 0 \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

$$\|A\|_\infty = \max\left\{ \left|\tfrac{1}{3}\right| + \left|-\tfrac{1}{3}\right| + \left|\tfrac{1}{4}\right|, \right.$$

$$\left|-\tfrac{2}{3}\right| + \left|\tfrac{1}{5}\right| + |0|,$$

$$\left. \left|-\tfrac{1}{2}\right| + \left|\tfrac{1}{4}\right| + \left|\tfrac{1}{5}\right| \right\}$$

$$= \max\left\{ \frac{11}{12}, \frac{13}{15}, \frac{19}{20} \right\} = \frac{19}{20} < 1$$

$\underline{\text{Thm}} \Rightarrow \quad \rho(A) \le \|A\|_\infty = \frac{19}{20} < 1.$

So all $|\lambda_1|, |\lambda_2|, |\lambda_3| < 1$

$$\Rightarrow \quad A^k \longrightarrow 0 \quad \text{as } k \to \infty.$$

$$u^{(k)} \longrightarrow 0 \quad \text{as } k \to \infty$$

L^2 norm on A.

$$\|A\|_2 = \max \{ \|Au\|_2 \mid \|u\|_2 = 1 \}$$

<u>Prop</u> Let the largest singular value

$\sigma_1 < 1$.

Then $\rho(A) \le \|A\|_2 < 1$

so $A^k \to 0$