Last time ...

- If T is a matrix, and all eigenvalues λ_i have $|\lambda_i| < 1$,

  then $T^k \to 0 \iff u^{(k)} \to 0$.

- If T has one eigenvalue $\lambda = 1$ w/ no repeats

  then $u^{(k)} \to u^*$

  where u^* is a fixed point of T.

- If the $\|T\|_\infty = \underset{\text{row sum}}{\text{max absolute}} < 1$

  $\Rightarrow T^k \to 0$. and

  $u^{(k)} \to 0$.

The converse of this result is not true!

$$T = \begin{pmatrix} 1 & -\frac{1}{3} & -2 \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

We show that $T^k \longrightarrow 0$ as $k \to \infty$ since

$$|\lambda_i| < 1$$

but

$$\|T\|_\infty = \max \left\{ \begin{array}{l} 1 + \frac{1}{3} + 2 = \frac{10}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{array} \right\}$$

$$= 10/3 > 1.$$

If $\|T\|_\infty \geq 1$ we learned nothing about how T^k behaves.

§ 9.3 Markov Processes

Word Problem!

Suppose if today is snowing, tomorrow has a 70% chance of also snowing. But if there's no snow, there's an 80% chance there's no snow tomorrow. If today is snowing, what's the probability of it snowing in 7 days?

We can turn this into a linear iterative system!

Suppose $u^{(k)} = \begin{pmatrix} s^{(k)} \\ n^{(k)} \end{pmatrix}$

where $s^{(k)}$ is the probability that it snows on k^{th} day.

$n^{(k)}$ ... no snow on k^{th} day.

$$s^{(k)} + n^{(k)} = 1 \quad \forall k.$$

$$s^{(k+1)} = 0.7 s^{(k)} + 0.2 n^{(k)}$$

$$n^{(k+1)} = 0.3 s^{(k)} + 0.8 n^{(k)}$$

$$u^{(k+1)} = \begin{pmatrix} s^{(k+1)} \\ n^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} s^{(k)} \\ n^{(k)} \end{pmatrix}$$

$$= J u^{(k)}$$

$$u^{(k+1)} = \begin{pmatrix} s^{(k+1)} \\ n^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} s^{(k)} \\ n^{(k)} \end{pmatrix}$$

$0.7 + 0.3 = 1$ $\quad$ $0.2 + 0.8 = 1$

$$= T u^{(k)}$$

$$u^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ since it's snowing today

$$u^{(7)} = T^7 u^{(0)} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}^7 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.405 \\ 0.595 \end{pmatrix}$$

40.5 % chance of snow in 7 days!

The limit $u^{(k)}$ as $k \to \infty$ encodes the average probabilities of snow or no snow.

As $k \to \infty$ $u^{(k)} \longrightarrow \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$

So on average 40% of the days snow in this model.

Theory behind this model.

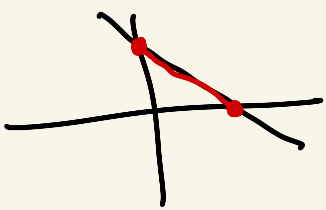
<u>Def</u> A vector $u = (u_1, u_2, \ldots, u_n)$ is a probability vector such that

$1 \geq u_i \geq 0$ and $u_1 + u_2 + \ldots + u_n = 1$.

For example $u = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$ is a prob. vector.

Side note!

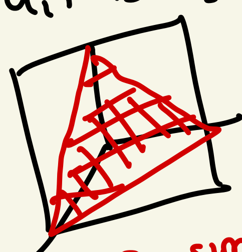
$u_1 + u_2 = 1$ 1-simplex



$(x+y=1)$

$u_1 + u_2 + u_3 = 1$

$n = (1,1,1)$



2-simplex

Def: A matrix T is a transition matrix if $1 \geq t_{ij} \geq 0$ and

$$\sum_{i=1}^{n} t_{ij} = 1 \quad \text{(columns add to 1)}.$$

$$T = \begin{pmatrix} 0.7 & 0.2 \\ \underline{0.3} & \underline{0.8} \end{pmatrix}$$

probability of it snowing tomorrow given that it snowed today

probability of weather given that there was no snow today

Prop If $u^{(k)}$ is a probability vector and T is a transition matrix, then $u^{(k+1)} = Tu^{(k)}$ is also a probability vector.

Pf: $u^{(k)}$ is a prob vector

iff $u^{(k)}_j > 0$ and

$(1,1,..1) \cdot u^{(k)} = 1$

T is a transition matrix

iff $t_{ij} > 0$ and

$(1\ 1\ ..\ 1)\, T = (1\ 1\ ...\ 1)$.

$(1,1,1 \ldots 1) \cdot u^{(k+1)}$

$= (1\ 1\ \ldots\ 1)(T u^{(k)})$

$= \underbrace{(1\ 1\ \ldots\ 1)\, T}\ u^{(k)}$

$= (1\ 1\ \ldots\ 1)\, u^{(k)} = 1$

$\Longrightarrow u^{(k+1)}$ is a probability vector.

Def We say T is a regular transition matrix if there is some integer $n > 0$ such that T^n has all strictly positive entries.

Ex $T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$ is regular

T^1 has all positive entries.

Ex $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ <u>not regular</u>.

Thm If T is a regular transition matrix. Then T has a eigenvalue $\lambda = 1$ which has no repeats and a unique probability eigenvector u^* for $\lambda = 1$.

(u^* represents average probabilities, <u>steady</u>
u^* is a fixed point, <u>fixed point</u>)

Given a probability vector $u^{(0)}$

$$u^{(k)} \longrightarrow u^* \quad \text{as} \quad k \to \infty.$$

Given a regular transition matrix,

the average probability vector

is always $\lim_{k\to\infty} u^{(k)} = u^*$.

$$T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$$

$\hookrightarrow$ $\lambda = 1$ (as predicted!)

$v = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$

$\hookrightarrow$ $\lambda = \frac{1}{2}$

$$\begin{pmatrix} 2/3 \\ 1 \end{pmatrix} \longrightarrow \frac{1}{1+\frac{2}{3}} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

average behavior as predicted!

Ex A taxi company (rideshare company?)

runs in the twin cities

If person boards in Mpls

$\longrightarrow$ 10% St paul

$\longrightarrow$ 30% suburbs

Boarded in St paul

$\longrightarrow$ 30% Mpls

$\longrightarrow$ 30% suburbs

Burbs

$\longrightarrow$ 40% Mpls

$\longrightarrow$ 30% St paul

Where are the taxis on average?

Build the transition matrix column by column.

$$T = \begin{array}{c} \\ \text{mpls} \\ \text{St.p.} \\ \text{burbs} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Mpls} & \text{St.P} & \text{burbs} \end{array} \\ \begin{pmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix} \end{array}$$

T is a regular transition matrix

$$t_{ij} > 0 \qquad \sum_{i=1}^{3} t_{ij} = 1$$

In fact $\lambda = 1, \ \lambda = 0.3, \ \lambda = 0$

$$v = \left(\frac{11}{7}, \frac{16}{21}, 1 \right)$$

$$u^* = \frac{1}{\frac{11}{7} + \frac{16}{21} + 1} \begin{pmatrix} \frac{11}{7} \\ \frac{16}{21} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.47\ldots \\ 0.22\ldots \\ 0.3\ldots \end{pmatrix}$$

$$u^* = \frac{1}{\frac{11}{7} + \frac{16}{21} + 1} \begin{pmatrix} \frac{11}{7} \\ \frac{16}{21} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.47\ldots \\ 0.22\ldots \\ 0.3\ldots \end{pmatrix}$$

average ~47% Mpls

(rounding...)

~22% St. P

~30% Suburbs

This vector is a fixed point for T!

$$\begin{pmatrix} 0.6 & 0.3 & 0.4 \\ 0.1 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix} \begin{pmatrix} 0.47\ldots \\ 0.22\ldots \\ 0.3\ldots \end{pmatrix}$$

$$= \begin{pmatrix} 0.47\ldots \\ 0.22\ldots \\ 0.3\ldots \end{pmatrix}$$