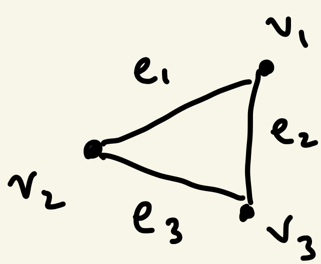
One use of Markov processes. -

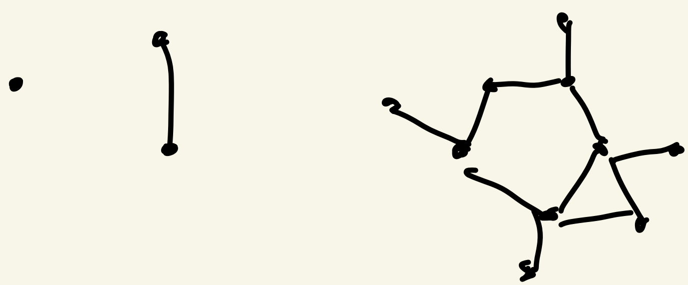
<u>Def</u> A graph is a collection of vertices and edges.

A vertex is represented by a dot, and an edge is represented as a line between two vertices.

<u>Ex</u>



There are 3 vertices $v_1 v_2 v_3$ and 3 edges $e_1 e_2 e_3$.

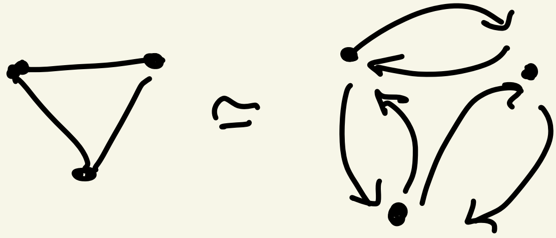


Relatedly, there's something called a digraph ...

It's a graph, but edges now have a direction, drawn as arrows $\longrightarrow$.

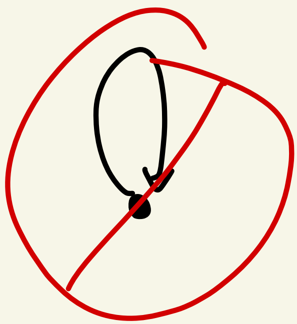
<u>Ex</u>

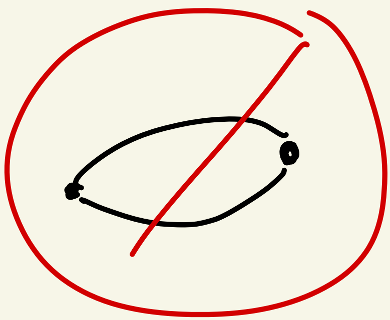
Sometimes a regular graph is called an "undirected graph".



A graph is undirected,

no edges from a vertex to itself and no two edges go between the same two vertices.





Recall If it snows today, then 70% chance it will snow tomorrow, no snow means an 80% chance of no snow tomorrow.

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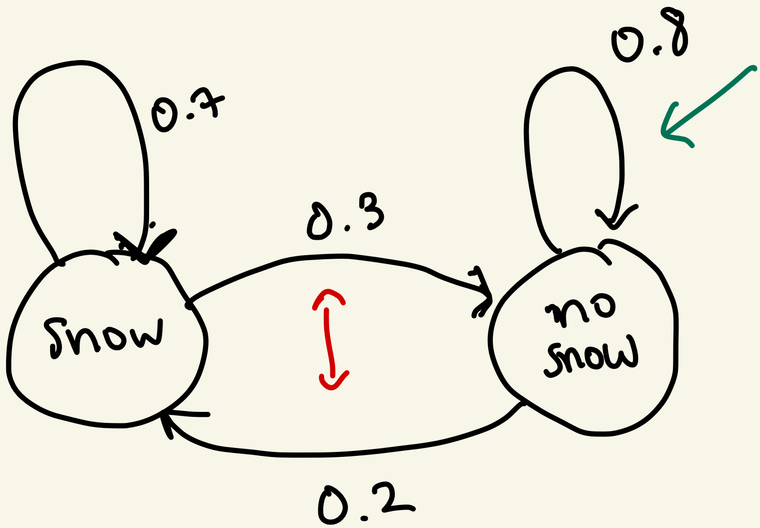


This is a weighted directed graph w/ self loops and multi edges

So the weather as it changes from day to day "walks" randomly along the graph.

What percentage of the time is the "walk" at each vertex?

$\Updownarrow$

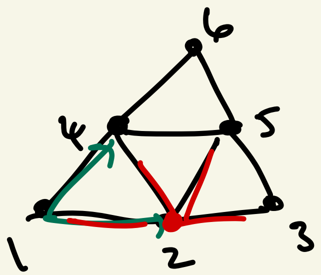
What's the weather on average?

Abstractify...

given a graph, and a "random walk" on the graph, what's the probability you'll be at each vertex? (2.6.6)

undirected, no self loops, no multi edges

Ex Given the graph, has 6 vertices and 9 edges.



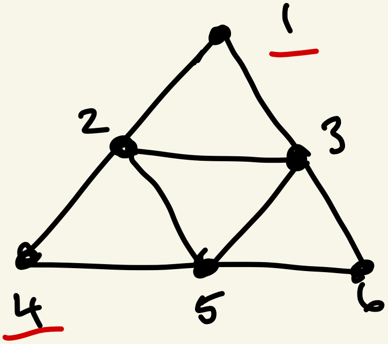
Suppose you are walking around on this graph randomly.

You are as likely to go along each edge at a given vertex.

What's the probability you'll be at any given vertex?

This is a Markov process!

$$T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{4} & & & & \\ \frac{1}{2} & 0 & & & & \\ 0 & \frac{1}{4} & & & & \\ \frac{1}{2} & \frac{1}{4} & & & & \\ 0 & \frac{1}{4} & & & & \\ 0 & 0 & & & & \end{pmatrix} \end{array}$$



Change vertex labels?

$$T = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

6×6 !!

Q: Is this a regular transition matrix?

A: Yes!

Claim: T^2 has all nonzero entries.

T^2 represents the probabilities of where you'll be after walking along 2 edges.

At any vertex, any other vertex is 2 edges away! T^2 has nonzero entries.

T has eigenvalue $\lambda = 1$

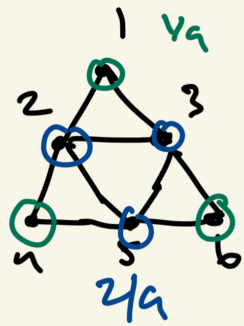
$$\left(\begin{array}{ll} \lambda = -\frac{1}{2} & 3 \text{ times} \\ \lambda = \frac{1}{4} & \text{repeat} \end{array} \right)$$

$\lambda = 1$ has eigenvector

$$u = (1, 2, 2, 1, 2, 1).$$

We can scale u to make it a probability eigenvector.

connected

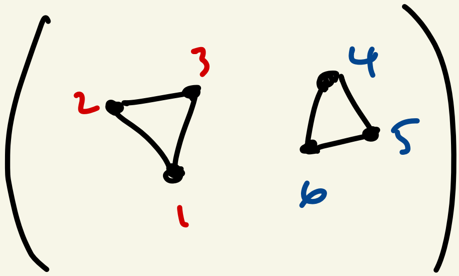


$$u^* = \frac{1}{1+2+2+1+2+1} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{array}{cc} \begin{pmatrix} 1/9 \\ 2/9 \\ 2/9 \\ 1/9 \\ 2/9 \\ 1/9 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Note: Consider the graph

not connected



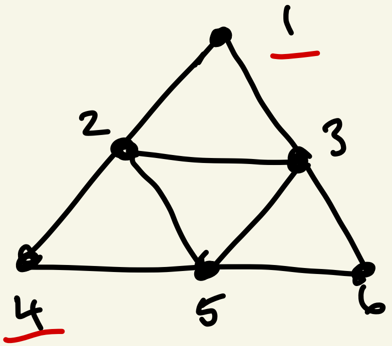
two disjoint triangles.

The transition matrix for a random walk on this graph is not regular!

T^k will always have zero entries,

There's never a path from vertex 1 to vertex 4. So the 41 entry of T^k is always 0.

Def: A graph is called connected if given two vertices v, w, there is a path from v to w.



what if this matrix represented Yes or No instead?

$$T = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

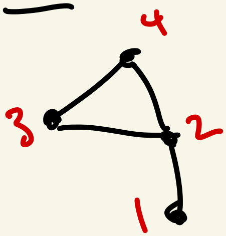
You can represent every graph as a matrix; 2 ways.

- Adjacency matrix
- Incidence matrix (2.6)

tomorrow

__Def__: Given a graph G w/ n vertices the adjacency matrix A is an $n \times n$ matrices such that

$$(A)_{ij} = \begin{cases} 0 & \text{if there is no edge } i - j \\ 1 & \text{if there is an edge between vertex } i \text{ and vertex } j. \end{cases}$$

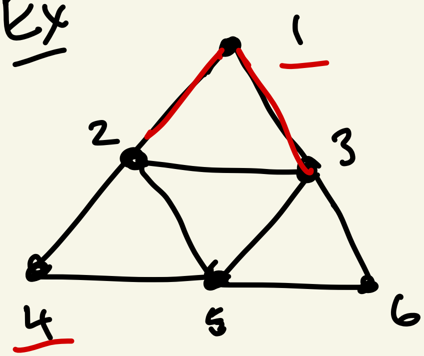
__Ex__: 

Find adjacency matrix

(A is symmetric btw)

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Ex



$$T = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} \color{red}{1} & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Since A is symmetric, the eigenvalues of A are real and can measure properties of the graph.