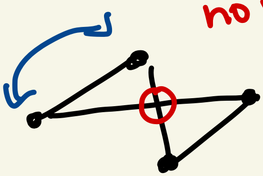
- Last day of new material
  for the course.

- HW 13 due tonight

- Final Exam Friday 7/31
  10:10am - 12:10 pm + 15 min
  to submit?

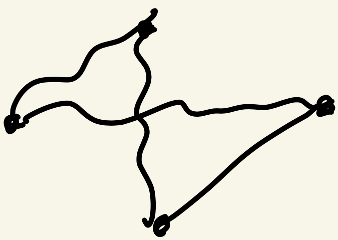
A graph is a collection of vertices and edges, vertices are dots while edges are lines.

no vertex

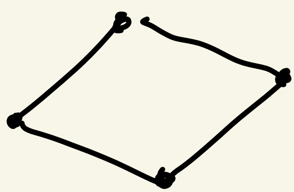


4 vertices

4 edges



Why is this the same graph?

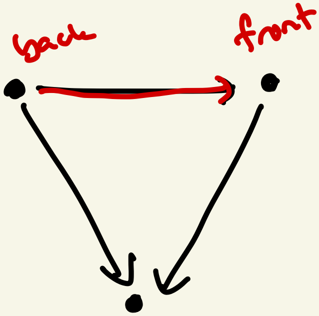


You can unflip the top vertices to see that the graph is a square.

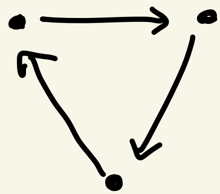
Digraph, directed graph.

Here, the edges are "directed".

We draw them as arrows.



$\neq$



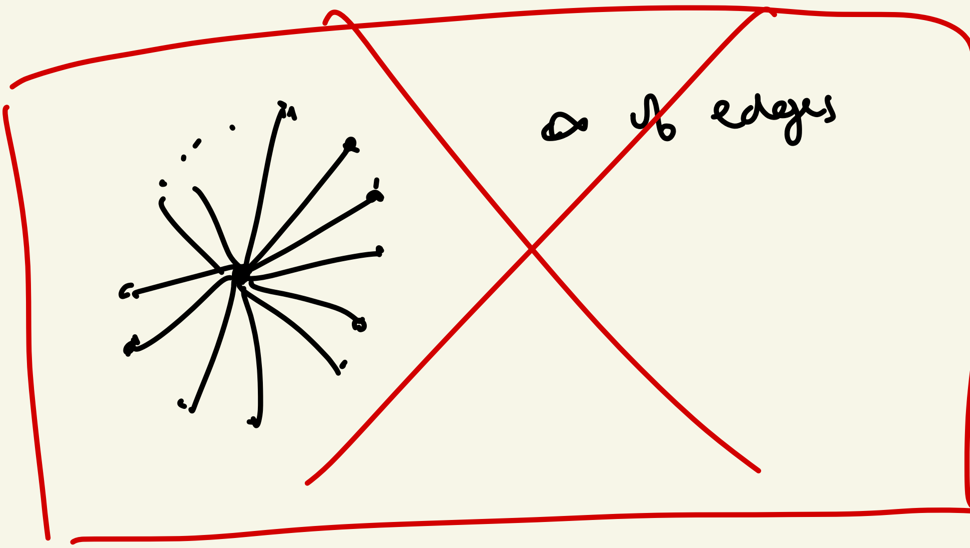
No "circuit".

This one has a "circuit"

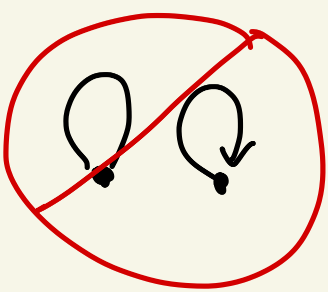
Rules

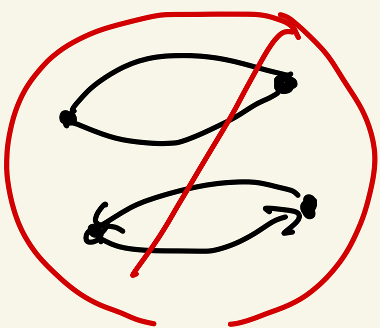
1. Graphs are finite.

   (No infinite number of vertices, no infinite number of edges)



2. Graph and digraphs have no self-loops or multi-edges

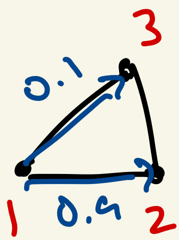




Associated are two matrices

- adjacency matrix,

matrix of 0's 1's which measured whether a vertex had an edge between them or not!



$$A_{adj} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \left[\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\right] & \\ 3 & & & \\ & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

$\rightsquigarrow$ T for a random Markov chain on the graph

G.

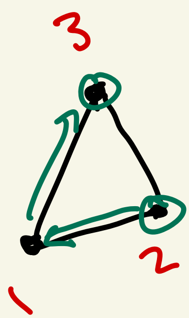
$$T = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$ is the translation matrix for the above graph.

Fact: Consider the adjacency matrix A of a graph G.

Then $(A^k)_{ij}$ is the # of paths of length k from vertex i to vertex j.

A^2 measure path of length 2 or chains of 2 edges

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \boxed{2} & 1 & 1 \\ 1 & 2 & \boxed{1} \\ 1 & \boxed{1} & 2 \end{bmatrix} \end{array}$$



Adjacency

$\downarrow$ + self multiedges

- Markov chains *
  - probability *

- CS

- Incidence Matrix

algebra

topology (shapes) without reference to angles or distances

<u>Def</u> Given a $\boxed{\text{directed}}$ G the incidence matrix associated to G is a $m \times n$ matrix

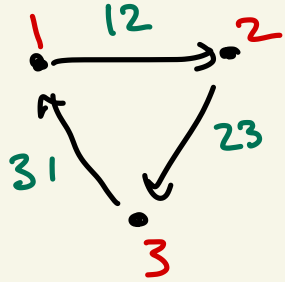
where $m =$ # of edges
$n =$ # of vertices of G

(for every edge there is a row
for every vertex there's a column)

$$(A_{inc})_{ij} = \begin{cases} 0 & \text{if edge } i \text{ does not touch vertex } j \\ 1 & \text{if edge } i \text{ starts at vertex } j \\ -1 & \text{if edge } i \text{ ends at vertex } j \end{cases}$$

easier to work w/ but contains all data of G.

Ex

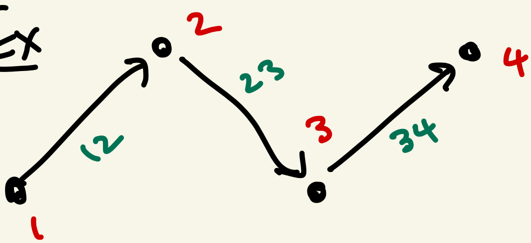


3 vertices

3 edges

$$A_{inc} = \begin{array}{c} \\ 12 \\ 23 \\ 31 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \end{array}$$

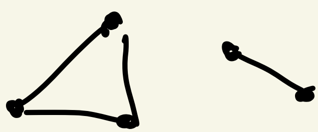
Ex



$$A_{inc} = \begin{array}{c} \\ 12 \\ 23 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

<u>Def</u> A graph (digraph) is connected if every pair of vertices can be connected by a series of edges (aka path).

Yesterday, Transition matrices for random walks on connected graphs are regular.

 is disconnected!

Prop If A is an incidence matrix for a connected digraph

then $\ker(A) = \text{span}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Pf: Suppose $z \in \ker(A)$.

Let $z = (z_1, z_2, \ldots, z_n)$.

We want to prove that $z_1 = z_2 = \ldots = z_n$ so that $z \in \text{span}((1, 1, \ldots, 1))$.

Since $z \in \ker(A)$, $Az = 0$.

Looking at one row of A ...

But $(A)_{i*} \cdot \vec{z} = 0$

But $(A)_{i*} \rightsquigarrow e$ some edge of the graph G

$$v_j \bullet \longrightarrow \bullet\, v_k$$

$$A_{i*} = \begin{array}{c} \begin{array}{cccccccc} & & j & & & k & & \end{array} \\ \left(\begin{array}{cccccccc} 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{array} \right) \end{array}$$

Since $(A)_{i*} \cdot z = 0$

$$\begin{array}{c} \begin{array}{cccccccccccc} & & j & & & & & k & & \end{array} \\ \left(0, 0, \underline{1}, 0\,0, \ldots, 0, \underline{-1}, 0, 0 \right) \end{array}$$

$$\cdot \left(z_1, z_2, \ldots, z_n \right)$$

$$= z_j - z_k = 0$$

$\Rightarrow$ $z_j = z_k$. when there is an edge $v_j \to v_k$.

What if there's no edge from

$v_1 \to v_2$ for example?

Is $z_1 = z_2$? Yes

Since G is connected

there's a path from

$v_1 \to v_{k_1} \to v_{k_2} \to \cdots \to v_2$

(finite)

$\Rightarrow \quad z_1 = z_{k_1} = z_{k_2} = \cdots = z_2.$

1 and 2 were arbitrary. any two vertices are connected by a path.

$\Rightarrow \quad z_1 = z_2 = z_3 = \cdots = z_n .$

$\Rightarrow \quad \vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$

$$\ker(A) \subseteq \operatorname{span}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$$

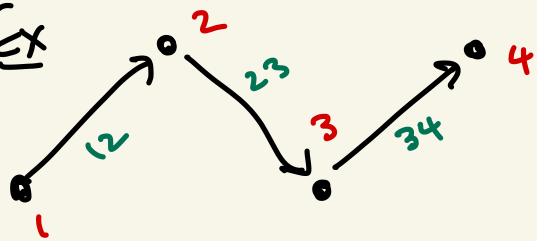
could still be $\ker(A) = 0$.

$$\text{But } \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} \in \ker(A)$$

$$\Rightarrow \quad \ker(A) = \operatorname{span}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.$$

$\square$

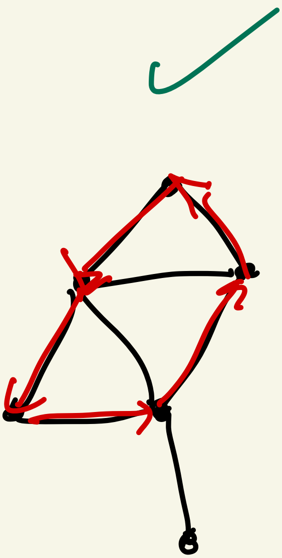
Ex

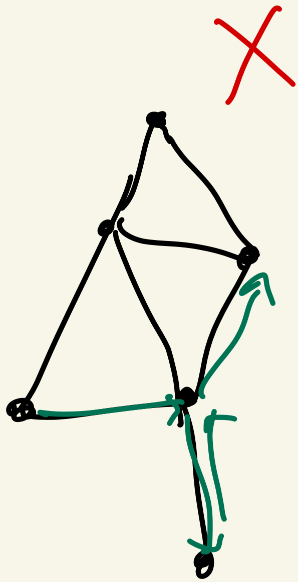


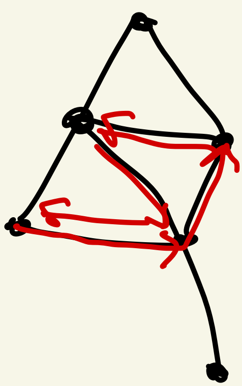
$$A_{inc} = \begin{array}{c} \\ 12 \\ 23 \\ 34 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

$$\ker(A) = \operatorname{span}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Def: A circuit in a graph is a sequence of edges which begins at the same vertex.







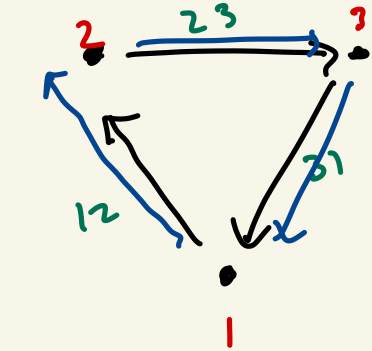
Suppose we have a directed graph w/ incidence matrix A_{inc}, we have a labeling and ordering of all the edges.

Then given a circuit
$C = e_1, e_2, \dots, e_k$.

we can associate it to a vector $m \times 1$

$$(v_c)_i = \begin{cases} 0 & \text{if edge } i \text{ is not involved in the circuit} \\ 1 & \text{if circuit goes forward along edge} \\ -1 & \text{if the circuit goes backwards along the arrow.} \end{cases}$$

Ex



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 12 & & & \\ 23 & & \left[\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{matrix}\right] & \\ 31 & & & \end{array}$$

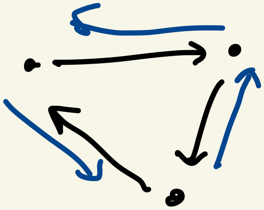
Circuit is labeled as

$\longrightarrow$ 12, 23, 31

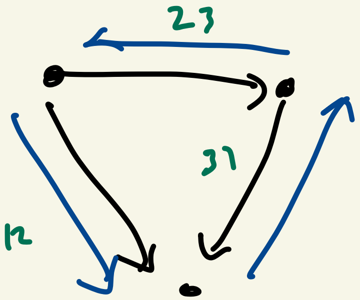
$\longrightarrow$ 1231

associated vector is

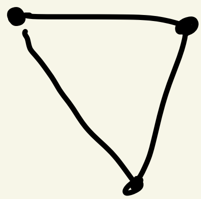
$$\longrightarrow \quad \begin{array}{c} 12 \\ 23 \\ 31 \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = v_c$$



$$v_c = \begin{array}{c} 12 \\ 23 \\ 31 \end{array} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

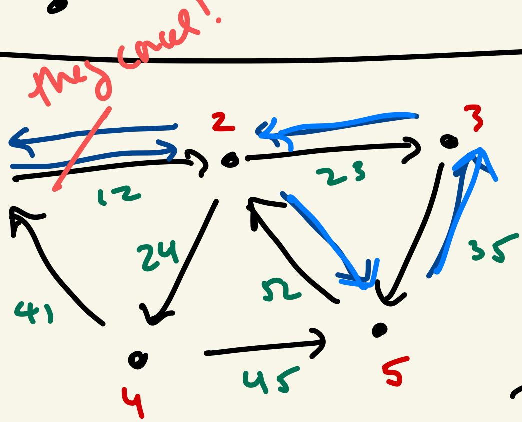


$$v_c = \begin{array}{c} 12 \\ 23 \\ 31 \end{array} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$



$\longrightarrow$ pick an path of arrows arbitrarily

Ex

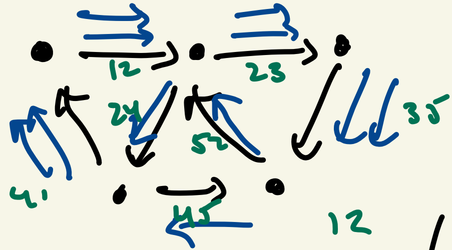


$$v_c = \begin{array}{c} 12 \\ 23 \\ 24 \\ 35 \\ 41 \\ 45 \\ 52 \end{array} \begin{pmatrix} 1-1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 12 & 1 & -1 & 0 & 0 & 0 \\ 23 & 0 & 1 & -1 & 0 & 0 \\ 24 & 0 & 1 & 0 & -1 & 0 \\ 35 & 0 & 0 & 1 & 0 & -1 \\ 41 & -1 & 0 & 0 & 1 & 0 \\ 45 & 0 & 0 & 0 & 1 & -1 \\ 52 & 0 & -1 & 0 & 0 & 1 \end{array}$$

into color of this thing!

$$= \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$



in the cokernel of

$$v_c = \begin{array}{c} 12 \\ 23 \\ 24 \\ 35 \\ 41 \\ 45 \\ 52 \end{array} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

= total number of traverses of that edge in the circuit!

<u>Thm</u> Each circuit C in a digraph G is represented by a vector v_c.

Moreover $v_c \in \operatorname{coker}(A_{inc})$.

In fact vectors of the form v_c generated the $\operatorname{coker}(A)$.

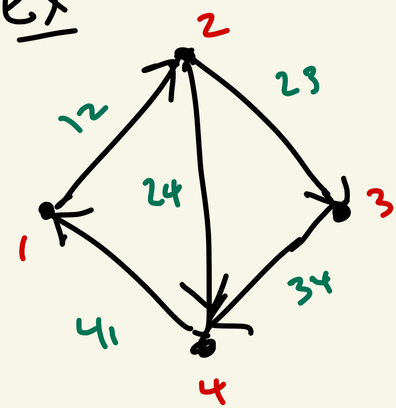
$\dim(\operatorname{coker}(A_{inc}))$ = # of independent circuits of G.

The $\operatorname{coker}(A_{inc}) = \operatorname{Span}(v_{c_1}, \ldots, v_{c_k})$

$c_1 \ldots c_k$ are the independent circuits.

All other circuits are combinations of these circuits!

<u>Ex</u>

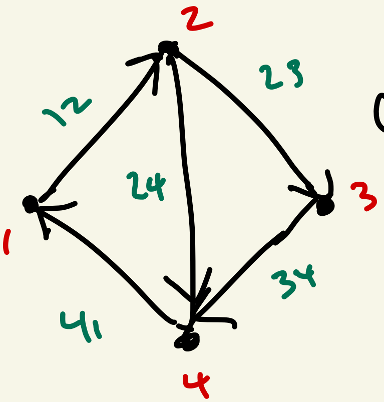


$$A_{inc} = \begin{array}{c} \\ 12 \\ 23 \\ 34 \\ 41 \\ 24 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

$$\operatorname{coker}(A_{inc}) = \ker(A^T)$$

$$= \ker \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix} \xrightarrow[\text{reduction}]{\text{row}}$$

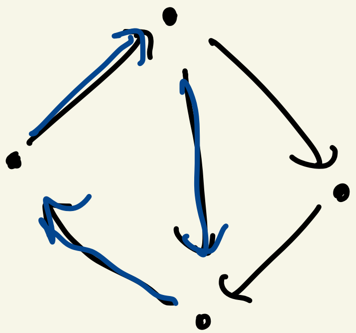
$$= \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$$



$$\operatorname{coker}(A_{inc}) = \ker(A^T)$$

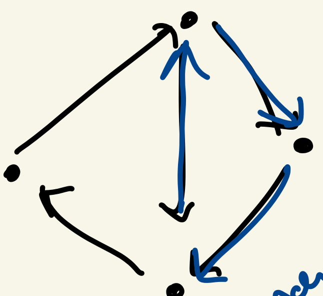
$$= \ker\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}$$

$$= \operatorname{Span}\underbrace{\left(\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\-1\end{pmatrix}\right)}_{\text{these correspond to circuits.}}$$



1 independent cycle

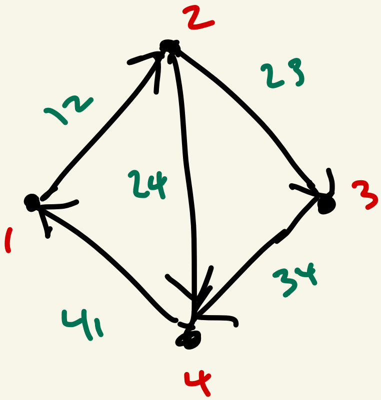
$$\begin{array}{c} 12 \\ 23 \\ 34 \\ 41 \\ 24 \end{array}\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}$$



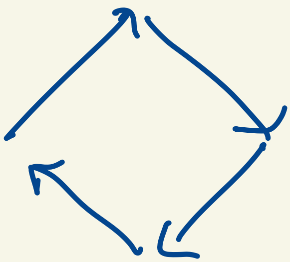
other independent cycle

$$\begin{array}{c} 12 \\ 23 \\ 34 \\ 41 \\ 24 \end{array}\begin{pmatrix}0\\1\\1\\0\\-1\end{pmatrix}$$

Claim: All other cycles are combinations of these two cycles!



$$\operatorname{coker}(A) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$$



$$v_c = \begin{array}{c} 12 \\ 23 \\ 34 \\ 41 \\ 24 \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

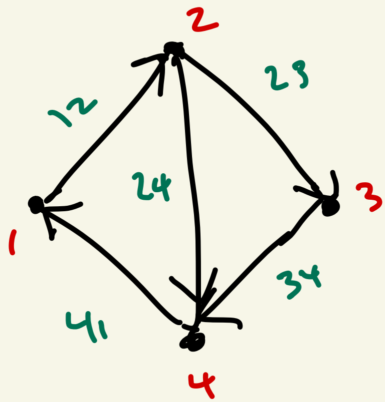
$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{coker}(A)$ since its a circuit

$c_1\ c_2$?

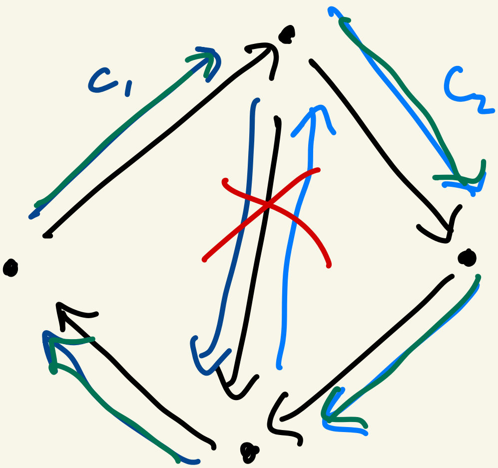
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$c_1 = c_2 = 1$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$



$$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix} = \underbrace{\begin{pmatrix}1\\0\\0\\1\\\boxed{1}\end{pmatrix}}_{v_{c_1}} + \underbrace{\begin{pmatrix}0\\1\\1\\0\\\boxed{-1}\end{pmatrix}}_{v_{c_2}}$$



Adding just means do 1 circuit then do the other.

$c_1 + c_2 =$ circuit around the whole square since $\downarrow\uparrow$ canceled.

or $$\begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix} + \begin{pmatrix}0\\1\\1\\0\\-1\end{pmatrix}$$

(§2.6)

Now that we understand that

$\mathrm{coker}(A_{inc}) \longleftrightarrow$ independent circuits

needs proof

but proof not on final

Thm (statement of thm on final)

Given a connected graph G,

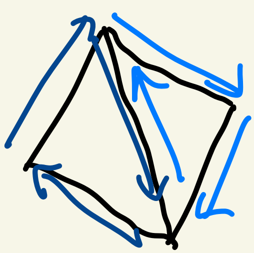
then

$$\underbrace{\#\text{vertices} - \#\text{edges}}_{\text{can be calculated purely by looking at } G} = \underbrace{1 - \#\text{independent circuits}}_{\text{data about the shape of the graph}}.$$

can be calculated purely by looking at G

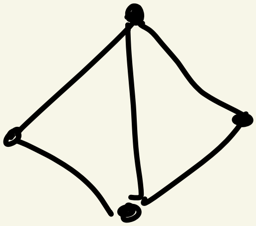
This is invariant when you keep the same # of holes.

data about the shape of the graph



2 independent circuits

This graph has "2 holes" in it.



2 = # of independent circuits

$\# v - \# e = 1 - \#$ ind. circ.

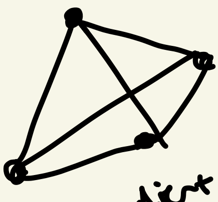
$4 - 5 = 1 - \#$ of ind circ.

$-1 = 1 - \#$

$\boxed{\# = 2}$

as predicted!

Ex



# of independent circuits

= 3

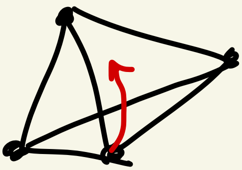
"4 holes"? Only 3 sides are independent!

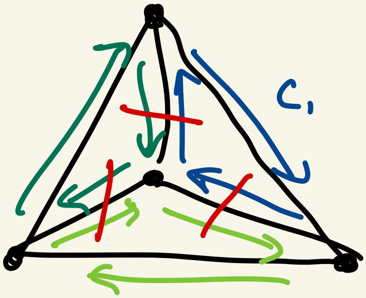
$\# v - \# e = 4 - 6$

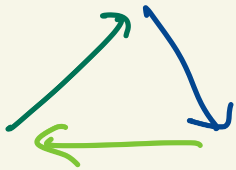
$= -2$

$-2 = 1 - \#$ of ind. circuits

view from the top.





$c_1 + c_2 + c_3 =$ 

Circuit corresponding to the 4th side.

So 4th depends on the other 3.

See Final Exam Review.

Use the cokernel of A_{inc}

<u>Thm</u> (statement of thm on final)

Given a connected graph G,

then

$\#$ vertices $-$ $\#$ edges $=$ $1 - \#$ independent circuits

<u>Pf</u>: Recall $\#$ of independent circuits $= \dim(\operatorname{coker}(A_{inc}))$.

by def. (every vector in $\operatorname{coker}(A)$ is a circuit vector)

A is $m \times n$ $\quad m = \#$ of edges
$n = \#$ of vertices

By 4 Fundamental subspaces

$$n = \operatorname{rank}(A) + \dim(\ker(A))$$

Rank-nullity

$$m = \operatorname{rank}(A^T) + \dim(\ker(A^T))$$

$$n = \operatorname{rank}(A) + \cancel{\dim(\ker(A))} \quad 1$$

$$\| $$

$$m = \operatorname{rank}(A^T) + \dim(\cancel{\operatorname{coker}}(A))$$

# of independent circuits.

$$\ker(A) = \operatorname{span}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\dim(\ker(A)) = 1$$

needs <u>proof</u> still

$$\operatorname{rank}(A_{inc}) = n - 1 = \#\text{ of vertices} - 1$$

$$\|$$

$$\operatorname{rank}(A^T_{inc}) = m - \#\text{ of indep. circuits}$$

$$= \#\text{ of edges.} - \#\text{ of ind. circ.}$$

$$n - 1 = m - \#\text{ of in circ}$$

$$n - m = 1 - \# \implies \#v - \#e = 1 - \#\text{ ind. circ.}$$

$\square$

Def Let G be a graph.

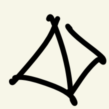
Then the Euler characteristic $\chi(G)$ of G

is defined by $\chi(G) = \#\text{vertices} - \#\text{edges}$

$$\overset{\text{Thm}}{=} 1 - \#\text{ind circuits}$$

$$= 1 - \dim(\operatorname{coker}(A_{inc}))$$

$\chi(G)$ depends on the shape of the graph, not on the exact graph.



$$\chi(G) = 1-2$$
$$= -1$$

$$\chi(G) = 4-5 = -1$$

Why does the cokernel of A_{inc}

$\longleftrightarrow$ circuits of G ?

A $m \times n$

$n \times m$

$$A^T_{inc} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

m = edges

n = vertices

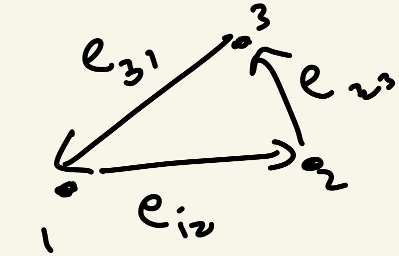
$\mathbb{R}^m$ has basis $\vec{e}_1, \dots \vec{e}_m$

= span(edge 1, edges 2, ... edge m)

$\mathbb{R}^n$ = arbitrary linear combs of $v_1 \dots v_n$

$$A^T_{inc}(e_{ij}) = v_i - v_j$$

$$v_i \xrightarrow{e_{ij}} v_j$$



$$A_{inc}^{T}(e_{12} + e_{23} + e_{31})$$

$$= v_1 - \cancel{v_2 + v_2} - \cancel{v_3 + v_3} - v_1$$

$$= 0$$ since it's a circuit all vertices cancel perfectly.

So $\text{coker}(A)$ = all circuit vectors.

For test

Euler characteristic formula $\#e - \#v = 1 - \#$ ind cir.

$\dim \text{coker} = \#$ of ind circuits