Review...

<u>Exam 2</u>

\#3. Compute an orthonormal basis for the orthogonal complement of the kernel of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 1 \\ -1 & 1 & 3 & 1 & 2 \end{pmatrix}.$$

<u>Def</u> A basis $\{u_1 \dots u_n\}$ is orthonormal if $u_i \cdot u_j = 0$ $\forall\, i \neq j$, and $\|u_i\| = 1$.

$\langle u_i, u_j \rangle = 0$

Gram-Schmidt is an algorithm that turns a basis into an orthogonal / orthonormal one.

Def Given a subspace $W \subseteq V$ in an inner product space.

The orthogonal complement $W^\perp$ is the subspace

$$W^\perp = \{ v \mid \langle v, w \rangle = 0 \ \forall w \in W \}.$$

"all the normal vectors to W".

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 1 \\ -1 & 1 & 3 & 1 & 2 \end{pmatrix}$$

Compute a orthonormal basis for $\ker(A)^\perp$.

We need any basis of $\ker(A)^\perp$, and then we can apply $\underbrace{\text{G-S.}}_{\text{last step.}}$

?

Recall: Given any matrix A,

then

$$\operatorname{img}(A) = \operatorname{coker}(A)^{\perp} = \ker(A^T)^{\perp}$$

$$\Leftrightarrow \left(\ker(A^T) = \operatorname{img}(A)^{\perp} \right)$$

and

$$\ker(A)^{\perp} = \operatorname{coimage}(A)$$

$$= \operatorname{img}(A^T).$$

So for this problem

$$\ker(A)^{\perp} = \operatorname{img}(A^T).$$

So we need a basis of $\operatorname{img}(A^T)$.

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 1 \\ -1 & 1 & 3 & 1 & 2 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 3 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\operatorname{img}(A^T) = ?$$

$\mathrm{Img}(A^T) = \mathrm{span}\{\text{columns of } A^T\}.$

$\Big(\mathrm{Img}(A) = \{ v = Ax \mid x \in \mathbb{R}^5 \}$

if $(x_1, x_2, x_3, x_4, x_5)$

then $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = A_1 x_1 + A_2 x_2 + \dots + A_5 x_5$

$= x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 + x_5 a_5$

where $A = (a_1 \ a_2 \ \dots \ a_5) \Big)$

$\underline{\ker(A)^\perp} \underset{\text{THM}}{=} \mathrm{img}(A^T) =$ span of columns of $\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 3 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}$

a basis of $\ker(A)^\perp$ is

$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}$?

So, this problem just asked for you to apply G-S to the rows of A

$$A = \begin{pmatrix} 1 & 2 & -1 & 3, & 1 \\ -1 & 1 & 3, & 1 & 2 \end{pmatrix}.$$

G-S $\quad v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}$

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} *$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = v_2 - \text{proj}_{w_1}^{v_2}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} - \frac{1 \cdot (-1) + 2 \cdot 1 - 3 + 3 + 2}{16} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

$$= \boxed{\frac{1}{16}} \begin{pmatrix} -19 \\ 10 \\ 51 \\ 7 \\ 29 \end{pmatrix} *$$

not need

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -19 \\ 10 \\ 51 \\ 7 \\ 29 \end{pmatrix}$$

$$u_1 = \frac{1}{4}\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{3952}}\begin{pmatrix} -19 \\ 10 \\ 51 \\ 7 \\ 29 \end{pmatrix}.$$

is an orthonormal basis for $\ker(A)^\perp$ = span rows

- remember that $\ker(A)^\perp = \operatorname{img}(A^T)$
- G-S
- unit vectors *

Exam 2 #5.

Find an inner product on $\mathbb{R}^2$ such that $(-1,2)$ and $(4,-2)$ are orthogonal in this inner product.

A formula $\langle x, y \rangle$ (\langle \rangle) is an inner product if

- bilinearity $\langle cx + dy, z \rangle$ factors etc
- symmetry $\langle x, y \rangle = \langle y, x \rangle$
- positivity $\langle x, x \rangle > 0$ $x \neq 0$

$\langle 0, 0 \rangle = 0.$

Any inner product on $\mathbb{R}^n$

$$\langle x, y \rangle = x^T K y$$

where K is an $n \times n$ positive definite matrix!

Find a 2x2 matrix K such that

$$\underbrace{(-1\ \ 2)\, K \begin{pmatrix}4\\-2\end{pmatrix}}_{\langle(-1,2),\,(4,-2)\rangle \,=\, 0} = 0.$$

$(-1,2) \perp (4,-2)$.

and K is positive definite.

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$b = c$ since K is symmetric

$$K = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$(-1\ \ 2)\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix}4\\-2\end{pmatrix} = 0$$

$$(-1\ \ 2)\begin{pmatrix} 4a-2b \\ 4b-2d \end{pmatrix} = 0$$

$$-4a+2b+8b-4d = 0$$

$$\underline{-4a+10b-4d = 0} \qquad \text{s.t. } K > 0.$$

$a = d \qquad -8a + 10b = 0$

$$b = \frac{4}{5}a$$

If $a = 5,\ b = 4$

$$K = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} > 0$$

it is pos def.

and

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 0$$

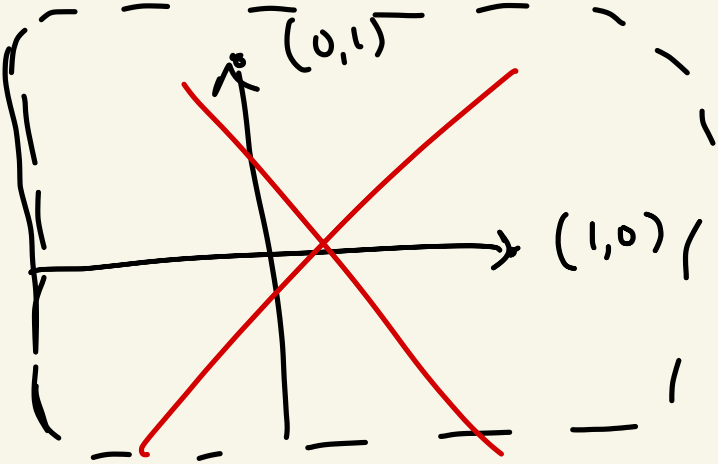
$\begin{pmatrix} -1 & 2 \end{pmatrix} K \begin{pmatrix} 4 \\ -2 \end{pmatrix} = 0 \qquad K > 0.$

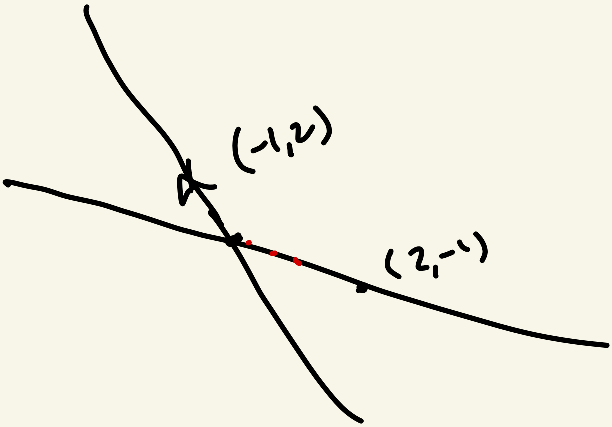
$(-1,2),\ (4,-2)$ form a basis of $\mathbb{R}^2$.

$(-1,2),\ (2,-1)$ form a basis in $\mathbb{R}^2$.

$v_1 = (-1,2) \quad v_2 = (2,-1)$.

We can write $\mathbb{R}^2$ in $v_1 v_2$ coordinates!





$$(x,y)_{v_1v_2} = x(-1,2) + y(2,-1)$$

$$(x,y)_{e_1e_2} = x(1,0) + y(0,1)$$

$$(x,y)_{v_1v_2} = x(-1,2) + y(2,-1) = \overset{\overset{S}{\shortparallel}}{\begin{pmatrix}-1 & 2\\ 2 & -1\end{pmatrix}}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\boxed{(x,y)_{v_1v_2} = S\,(x,y)_{e_1e_2}}$$

$(-1,2) = (1,0)_{v_1 v_2}$

$(4,-2) = (0,2)_{v_1 v_2}$

What if the inner we were looking for is just the dot product but in $v_1 v_2$ coordinates?

$$\langle (-1,2), (4,-2) \rangle = (1,0)_{v_1 v_2} \cdot (0,2)_{v_1 v_2}$$

$$\| \qquad = 0$$

$$\vec{x}^T K \vec{y} = \vec{x}_{v_1 v_2} \cdot \vec{y}_{v_1 v_2}$$

$$\boxed{S = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}}$$

$$\underline{\langle \vec{x}_{e_1 e_2}, \vec{y}_{e_1 e_2} \rangle}$$

$$= \langle S^{-1} \vec{x}_{v_1 v_2}, S^{-1} y_{v_1 v_2} \rangle$$

$$= (S^{-1} x)^T S^{-1} y = x^T \underbrace{(S^{-1})^T S^{-1}}\, y$$

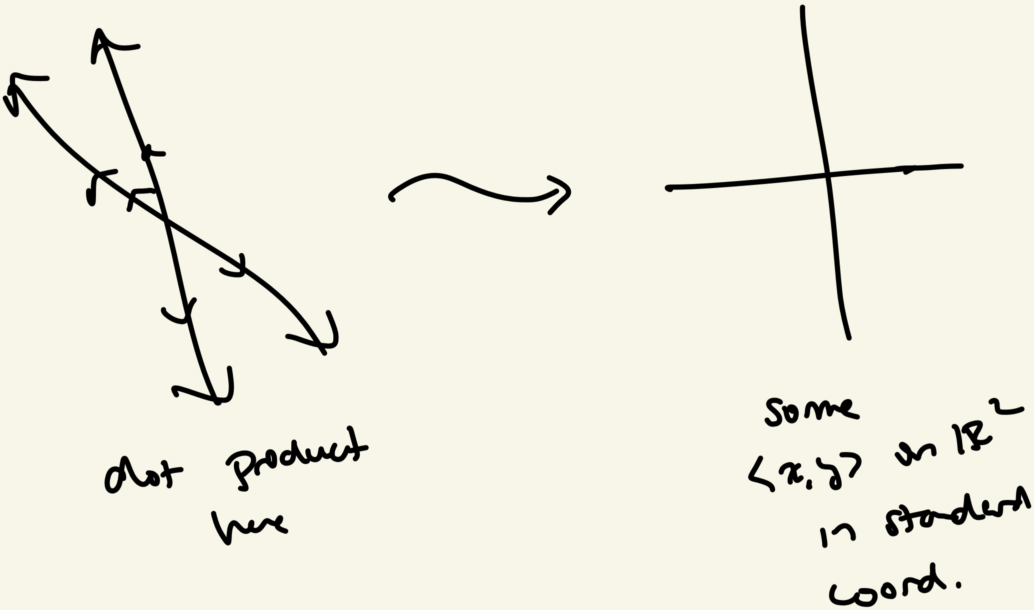
$K = (S^{-1})^T S^{-1}$

$$= x^T K y$$

$$K = (S^{-1})^T S^{-1} \qquad S = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$K = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}.$$



Change of basis for matrices

$$\vec{x}_{(v_i)} = \underline{S}\,\vec{x}$$

$\{v_1 \ldots v_n\}$ basis

$$S = (v_1 \ldots v_n)$$

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

$$B = S^{-1} A S$$

$$\begin{tikzcd}
{} \arrow[r, "B"] \arrow[d, "S"'] & {} \\
{} \arrow[r, "A"'] & {} \arrow[u, "S^{-1}"']
\end{tikzcd}$$