How do the systems $Ax = b$ and $A^T x = b'$ relate?

In general A is $m \times n$, so A^T is $n \times m$ so their kernels and images might lie in different vector spaces.

$$
\begin{array}{cccc}
A : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\
m \times n & n \times 1 & & m \times 1 \\
 & \cup & & \cup \\
 & \boxed{\ker(A)} \ * & & \boxed{\operatorname{img}(A)} \ *
\end{array}
$$

$$
\begin{array}{cccc}
A^T : & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\
n \times m & m \times 1 & & n \times 1 \\
 & \cup & & \cup \\
 & \boxed{\ker(A^T)} \ * & & \boxed{\operatorname{img}(A^T)} \ *
\end{array}
$$

Thm

let A be a $m \times n$ matrix.

Then $\text{rank}(A) = \text{rank}(A^T)$.

Pf

$\text{rank}(A) =$ dim (span of columns of A)

$=$ # of pivots in the reduced row echelon form

$=$ # of leading 1's

Recall, every leading 1 corresponded to an independent column of A

But every leading 1 had it's own row!

$$\begin{pmatrix} \boxed{1} & * & * & * & 0 & * & * & * & 0 & * & * & * \\ & & & & \boxed{1} & * & * & * & 0 & * & * & * \\ & & & & & & & & \boxed{1} & * & * & * \end{pmatrix}$$

$$\begin{aligned}
\operatorname{rank}(A) &= \dim(\text{span columns}) \\
&= \#\text{ of independent columns} \\
&= \#\text{ leading } 1\text{'s} \\
&= \#\text{ nonzero rows in RREF} \\
&= \dim(\text{span rows of } A) \\
&= \operatorname{rank}(A^T). \qquad \square
\end{aligned}$$

It's true

Thm Let A be an $m \times n$ matrix, and let A' be a matrix that can obtained from A by row operations.

$$A \xrightarrow[\text{row oper.}]{} A'$$

Then $\text{span}(\text{rows of } A)$
$= \text{span}(\text{rows of } A')$.

$\dim(\text{span}(\text{columns of } A))$
$= \dim(\text{span}(\text{columns of } A'))$.

Row operations <u>preserve</u> the row space of A.

# of nonzero rows in RREF

the same

$$= \dim\big(\text{span}(\text{rows of RREF})\big)$$

$$= \dim\big(\text{span}(\text{rows of } A)\big)$$

$$= \dim\big(\text{span}(\text{columns of } A^T)\big)$$

$$= \text{rank}(A^T).$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \end{pmatrix}$$

Add $r_1 + r_2 = r_3$

$r_2 \neq c r_1$

rank $(A) = 2$ since 2 independent rows

$= \#$ of independent columns

$$\begin{pmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Def Let A be an $m \times n$ matrix.

Define cokernel of A, $\mathrm{coker}(A)$,

to be

$$\mathrm{coker}(A) = \ker(A^T) \subseteq \mathbb{R}^m$$

Define the coimage of A, $\mathrm{coimg}(A)$,

$$\mathrm{coimg}(A) = \mathrm{img}(A^T) \subseteq \mathbb{R}^n.$$

These are subspaces.

Prop

$$\underbrace{\dim(\mathrm{img}(A))}_{=\ \mathrm{rank}(A)} = \underbrace{\dim(\mathrm{coimg}(A))}_{=\ \mathrm{rank}(A^T)}$$

$A^{m \times n}$

$\boxed{\ker(A)}$? $\boxed{\operatorname{img}(A)}$?

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$A^T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$\boxed{\operatorname{coker}(A)}$ $\boxed{\operatorname{coimg}(A)}$

<u>Thm</u> $\ker(A) = \operatorname{coimg}(A)^{\perp} \subseteq \mathbb{R}^n$

$\operatorname{coker}(A) = \operatorname{img}(A)^{\perp} \subseteq \mathbb{R}^m$

<u>Pf</u> In order to show $\ker(A) = \operatorname{coimg}(A)^{\perp}$ are equal, we can show that $x \in \ker(A)$ iff $x \perp w$ for all $w \in \operatorname{coimg}(A)$.

Let $x \in \text{coimg}(A)^{\perp}$

$\iff$ $x \perp$ all rows of A.

$\iff$ $a_{*j} \cdot x = 0$ $\forall$ rows

$*$ is fixed

$\iff$ $Ax = \vec{0}$

$\iff$ $x \in \ker(A)$.

Therefore $\text{coimg}(A)^{\perp} = \ker(A)$.

$$\begin{aligned}
\text{coker}(A) &= \ker(A^T) \\
&= \text{coimg}(A^T)^{\perp} \\
&= \text{img}((A^T)^T)^{\perp} \\
&= \text{img}(A)^{\perp}. \quad \square
\end{aligned}$$

We've actually done this already.

Let $A^T = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$

Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} \right\}$.

Find $W^\perp$

Method is to put $\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$

in the rows of a matrix.

Then $W^\perp = \ker(A^T)$

$= \text{coker}(A)$

$= \text{img}(A)^\perp$.

$A^T \longrightarrow \begin{pmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & 3 & 4 \end{pmatrix}$

$\ker(A^T) = \text{Span}\begin{pmatrix} -5 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -4 \\ 0 \\ 1 \end{pmatrix}$

$= \text{Span}\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}^\perp$

Prop Consider a system of equations $Ax = b$. Then this has a solution iff $b \perp \operatorname{coker}(A)$.

Pf: Recall that $Ax = b$ has no solution if $(A \mid b)$ you can get an inconsistent system.

$$\left(\begin{array}{cccc|c} & & & & \\ 0 & 0 & 0 & 0 & * \end{array} \right)$$

nonzero.

But more simply $\quad \text{Img}(A) = \{Ax \mid x \in \mathbb{R}^n\}$

$Ax = b$ has <u>no</u> solution if

$b \notin \text{img}(A)$

$b \notin$ span of columns of A.

$$\left(\begin{array}{l} \exists \text{ no } x \text{ s.t. } Ax = b \\ a_1 x_1 + \dots + a_n x_n \neq b \end{array} \right)$$

$b \notin \text{coker}(A)^\perp = \text{img}(A)$.

$Ax = b$ has a sol'n iff

$b \in \text{coker}(A)^\perp$

i.e. $b \perp \ker(A^T)$.

$\square$

Example Calculation of $\operatorname{coker}(A) = \ker(A^T)$.

Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix}$

We know that $\operatorname{coker}(A)$

$= \operatorname{img}(A)^{\perp}$.

When is $b \in \operatorname{img}(A)$?

Take $(A \mid \vec{b})$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ -1 & 1 & -1 & b_2 \\ 1 & -2 & 3 & b_3 \end{array}\right)$$

$$\xrightarrow[\text{reduce}]{\text{row}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & \boxed{b_3 + 2b_2 + b_1} \end{array}\right)$$

needs to be 0

This system is consistent

(aka $b \in \operatorname{img}(A)$)

iff $b_1 + 2b_2 + b_3 = 0$

aka

$\boxed{(1,2,1)} \cdot (b_1\ b_2\ b_3) = 0$

$\longrightarrow \operatorname{coker}(A) = \operatorname{img}(A)^{\perp}$

$= \operatorname{span}\begin{pmatrix}1\\2\\1\end{pmatrix}.$

Because $b \in \operatorname{img}(A)$

$\iff (A \mid b)$ consistent

$\iff b_1 + 2b_2 + b_3 = 0$

$\iff b \perp (1,2,1)$

$\iff \operatorname{coker}(A) = \operatorname{span}\begin{pmatrix}1\\2\\1\end{pmatrix}.$

Conceptually, coker(A) is conditions on the vector $\vec{b}$ which makes $Ax=b$ consistent.

<u>Thm</u> Let $b \in \text{img}(A)$. Then $\exists$ a unique w such that

$Aw = b$, and $w \in \text{coimg}(A)$
$= \ker(A)^{\perp}$.

Furthermore $\|w\|$ is minimal among solutions to the system of equations $Ax=b$.

If we wanted to find the unique smallest solution to $Ax=b$, it's the solution in the $\text{coimg}(A)$
$= \ker(A)^{\perp}$.

Let $\underline{x}$ be <u>any sol'n.</u>

$\ker(A)^{\perp} = \text{coimg}(A)$.

$x = w + z$, $z \in \ker(A)$

$w \in \text{coimg}(A)$.

$Ax = b$

$A(w+z) = b$

$Aw + \cancel{Az}^{\,0} = b$ $\quad z \in \ker(A)$

$Aw = b$, w is a sol'n

$w \in \text{coimg}(A)$.

It's unique by decomposition and $\|w\|$ is minimal by Δ inequality.

Full proof in book.

$V = \mathbb{R}$

$Y = \mathbb{Q}$

$y_1 = 3$

$y_2 = 3.1$

$y_3 = 3.14$

$y_4 = 3.141$

$\vdots$ Cauchy in $\mathbb{Q}$

Y closed iff Y contains a limit points

(y_i) be a sequence which converges in V $y_i \to y$, $y \in V$.

Suffices to show $y \in Y$.

Since $y_i \longrightarrow y$ in V

(y_i) is Cauchy.

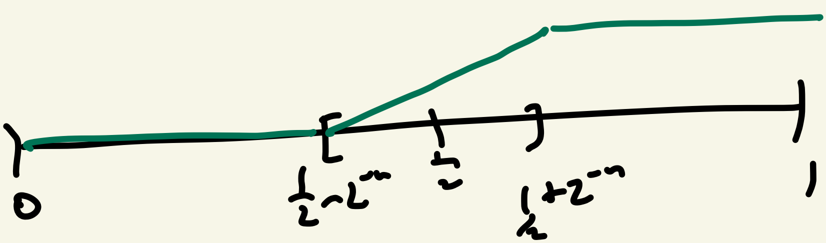
$\Rightarrow (y_i)$ is Cauchy in Y $\Big\}$

$\Rightarrow y \in Y$.

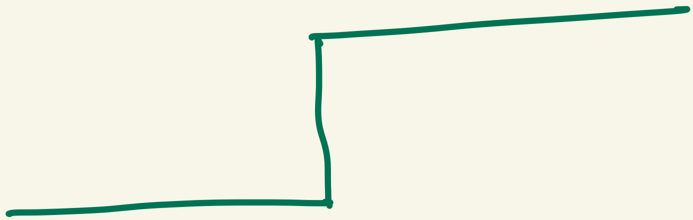
Y Banach

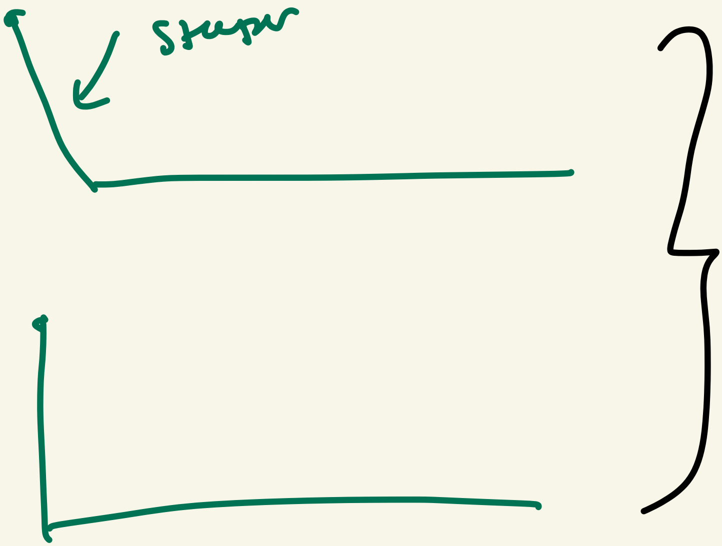
$C^0[0,1]$

$$f_n = \begin{cases} 0 & [0, \tfrac{1}{2} - 2^{-n}] \\ 1 & [\tfrac{1}{2} + 2^{-n}, 1] \\ \text{line} & [\tfrac{1}{2} - 2^{-n}, \tfrac{1}{2} + 2^{-n}] \end{cases}$$



Converges to





Thm If $f_1, f_2, f_3 \to f$ converges uniformly, then f is cts.

$\forall \varepsilon > 0 \ \exists N$ s.t. $\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon \quad \forall\, x \in [a,b]$.

$$t^3 = a_0 P_0 + a_1 P_1 + a_2 P_2 + a_3 P_3$$

Viewing $P_0, P_1, P_2, P_3 \in C^0[-1,1]$

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

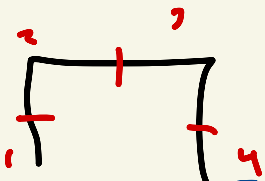
$$\|P_0\|^2 = \int_{-1}^{1} P_0^2 \, dx \qquad P_0 = 1$$

$$\|1\|^2 = \int_{-1}^{1} 1^2 \, dx = 2$$

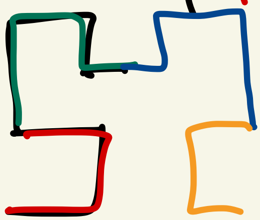
$f : [0,1] \to \mathbb{R}^2$ $\qquad f_1, f_2 \to f$

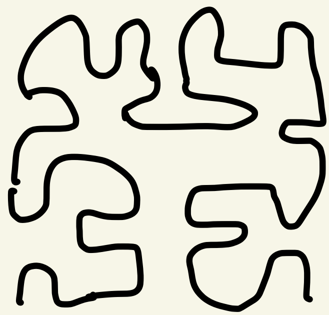
$\operatorname{img}(f) = [0,1]^2$, f cts

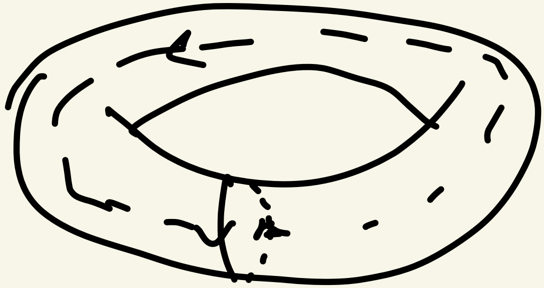
f_1 :



f_2 :







Loop : cts function

$[0,1) \longrightarrow X$