Final tomorrow! 10:10am - 12:10pm

+15 minutes to upload.

Ch. 7 Linearity

A function $T: V \to W$ is called a <u>linear transformation</u> if

$$T(v+w) = T(v) + T(w)$$

$$T(cv) = cT(v)$$

V, W any vector spaces

Ex $\frac{d}{dx}: C^1 \longrightarrow C^0$

differentiable functions s.t. f' is cts

cts functions

Ex $A \in M_{m\times n}(\mathbb{R})$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longrightarrow Ax.$$

In fact all linear transformations

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are just

matrix multiplication

$T(x) = Ax$ for some matrix A.

$$\mathrm{Hom}(V,W) = \{\text{all linear transformations } V \longrightarrow W\}$$

vector space

v.s of all $T : \mathbb{R}^n \to \mathbb{R}^m$

$$\dim(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))$$
$$= \dim(M_{m\times n}(\mathbb{R}))$$
$$= mn \qquad (\text{ex})$$

Given a vector space V, the dual space

$$V^* = \mathrm{Hom}(V, \mathbb{R})$$

$=$ all linear functions $V \longrightarrow \mathbb{R}$

$$(\mathbb{R}^n)^* = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$$

$=$ all linear functions $\mathbb{R}^n \longrightarrow \mathbb{R}^1$

$=$ all $1 \times n$ matrices

$=$ all row vectors $(a_1\ a_2\ \cdots\ a_n)$

(7.1)

A is a linear transformation

$\in M_{m \times n}(\mathbb{R})$

and so is a differential operator

Ex $D(u) = u'' - u$ is linear

$D(u) = u''' + u'' - u'$ is linear

$$D : C^2([a,b]) \longrightarrow C^0([a,b])$$
$$u \longmapsto u'' - u$$

$Ax = b \quad \Longleftrightarrow \quad D(u) = f$

$u'' - u = f$

Same principles apply!

Superposition principle.

Given any linear transformation

$T: V \to W$, then the

equation $\boxed{T(v) = w}$ $\rightarrow$ $Ax = b$

$u'' - u = f$

has solution

$$v = v^* + z$$

where v^* is one particular solution

$z \in \ker(T)$. $\quad (T(z) = 0)$.

① Find v^*, one particular sol.

② Find $\ker(T)$

③ All solutions are $v = v^* + \ker(T)$.

Ex

$$\begin{pmatrix} 2 & 1 & 4 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A \quad \vec{x} \quad = \quad \vec{b}$$

$$\vec{x} = \underline{v^*} + \underline{\ker(A)}$$

Row reduction

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

z is free

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 7/5 & 0 \\ 0 & 1 & 6/5 & 1 \end{array}\right)$$

$$x + \frac{7}{5}z = 0$$

$$y + \frac{6}{5}z = 1$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{7}{5}z \\ -\frac{6}{5}z + 1 \\ z \end{pmatrix}$$

$$= \underbrace{\frac{-1}{5}\begin{pmatrix} 7 \\ 6 \\ -5 \end{pmatrix} z}_{\ker(A)} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{v^*\text{, particular solution}}$$

Ex $\quad D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$

$$D(u) = u' - u \text{ ,}$$ this is a linear operator

$$D(u_1 + u_2) = D(u_1) + D(u_2)$$

$$D(cu_1) = c D(u_1)$$

$$u' - u = x - 3$$

$$D(u) = f$$

$$u = u^* + \ker(D)$$

To find u^*, guess that $u^* = ax + b$

$$(ax+b)' - (ax+b) = x - 3$$

$$a - ax - b = x - 3$$

$$-a = 1 \qquad a = -1$$

$$a - b = -3 \qquad b = 2$$

So the particular solution

is $u^* = -x+2$.

Now $u = -x+2 + \underbrace{\ker(D)}_{\text{How?}}$

$$\ker(D) = \{u \mid D(u) = 0\}$$

$$D(u) = u' - u = 0$$

Guess e^{rx}

$$re^{rx} - e^{rx} = 0$$

$$\cancel{e^{rx}}(r-1) = 0$$

$$r = 1$$

$$\ker(D) = \operatorname{span}(e^x) = ce^x$$

$$u = \underbrace{ce^x}_{\ker(D)} \underbrace{-x+2}_{\text{particular solution}}$$

Ch. 5 Minimization (App Stats)

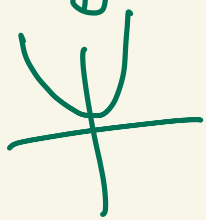
· minimize quadratic equations

$$q(x) = \underbrace{x^T K x}_{\substack{\text{quadratic} \\ \text{degree 2 terms}}} \underbrace{- 2x^T f}_{\substack{\text{linear} \\ \text{deg 1 terms}}} + c$$

The minimal value occurs at

$$x^* = K^{-1} f$$

when K is positive definite.



$$q(x,y) = \overbrace{2x^2 + 2xy + y^2}^{x^T K x} \overbrace{- 3x + 2y}^{-2x^T f} + 1 \qquad x = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find minimal value

$$K = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

K is pos definite

$\lambda > 0$

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$= (2-\lambda)(1-\lambda) - 1$$

$$= 2 - 3\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 3\lambda + 1$$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2} > 0$$

So K is pos definite.

Thm K is pos. def iff K has positive real eigenvalues. *

Thm $K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is pos. def. iff $a > 0$ $ad - bc > 0$.

$$
\begin{aligned}
-3x+2y &= -2\left(\tfrac{3}{2}x-y\right) \\
&= -2\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3/2 \\ -1 \end{pmatrix} \\
&\ -2\, x^T \boxed{f}
\end{aligned}
$$

$$
q(x) = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 2\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3/2 \\ -1 \end{pmatrix} + 1
$$

$$
x^* = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3/2 \\ -1 \end{pmatrix}
$$

$$
\begin{aligned}
x^* &= \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3/2 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3/2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -7/2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 5 \\ -7 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
q(x^*) &= c - f^T x^* = 1 - \begin{pmatrix} 3/2 & -1 \end{pmatrix}\begin{pmatrix} 5/2 \\ -7/2 \end{pmatrix} \\
&= 1 - \left(\frac{15}{4} + \frac{7}{2}\right) = \frac{4}{4} - \frac{15}{4} - \frac{14}{4} \\
&= \boxed{-\frac{25}{4}}
\end{aligned}
$$

min value of q.

- minimizing distance between subspace W to vector b.

$W = \text{img}(A)$

Take a basis of W and put them in the columns of a matrix A.

$\|Ax - b\|^2$ minimize

$K = A^T A$

$f = A^T b$

$\Rightarrow x^* = (A^T A)^{-1} A^T b$ least squares solution to $Ax = b$.

#3 Review

$$\overset{A}{\begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix}} \overset{x}{\begin{pmatrix} x \\ y \end{pmatrix}} = \overset{b}{\begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}}$$

Find least squares solution

$$x^* = \underbrace{(A^T A)^{-1}}\underbrace{A^T b}$$

$$A^TA = \begin{pmatrix} 0 & -3 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 1 \\ 1 & 6 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{77} \begin{pmatrix} 6 & -1 \\ -1 & 13 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 0 & -3 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$x^* = \frac{1}{77} \begin{pmatrix} 6 & -1 \\ -1 & 13 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{77} \begin{pmatrix} 1 \\ -13 \end{pmatrix}$$

Which vector from $\text{img}(A)$ is actually closest to $b = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$?

$$w^* = Ax^* = A(A^TA)^{-1}A^Tb$$

$$= \begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix} \frac{1}{77} \begin{pmatrix} 1 \\ -13 \end{pmatrix}$$

$$= \frac{1}{77} \begin{pmatrix} -13 \\ -16 \\ -24 \end{pmatrix}$$

Ch 9

Linear iterative systems

$u^{(0)}$, $u^{(k+1)} = T u^{(k)}$

where T is an $n \times n$ matrix

$$u^{(0)}, \quad \underset{\substack{\| \\ u^{(1)}}}{T^1 u^{(0)}}, \quad \underset{\substack{\| \\ u^{(2)}}}{T^2 u^{(0)}}, \quad \ldots$$

$$\Updownarrow$$

$$T, \; T^2, \; T^3, \; \ldots$$

Behavior depends on eigenvalues of T !

Compute the eigenvalues!

Thm The following are equivalent.

1. All eigenvalues of T have $|\lambda_i| < 1$ — T diagonalizable
2. $T^k \to 0$ (zero matrix) $k \to \infty$
3. $u^{(k)} \to 0$ (zero vector) $k \to \infty$ for any $u^{(0)}$.

#11. $T = \frac{1}{6}\begin{pmatrix} 4 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & -9 & 3 \end{pmatrix}$

$T^k \to 0$ or $u^{(k)} \to 0$

where $u^{(0)} = (1,0,1)$.

Find eigenvalues of T

Find eigenvalues of

$$\begin{pmatrix} 4 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & -9 & 3 \end{pmatrix}$$

then multiply by $\frac{1}{6}$ after.

$\lambda = 3,\ 3 \pm 3i$

$\lambda = \frac{1}{2},\ \frac{1}{2} \pm \frac{1}{2}i$

$\left|\frac{1}{2}\right| = \frac{1}{2}$ $\quad \left|\frac{1}{2} + \frac{1}{2}i\right| = \sqrt{\frac{1}{4} + \frac{1}{4}}$

$= \frac{\sqrt{2}}{2} < 1$

$\left|\frac{1}{2} - \frac{1}{2}i\right| = \frac{\sqrt{2}}{2} < 1.$

So $T^k \to 0$ and

$$u^{(k)} \to 0.$$

<u>Thm</u> The fixed points of a matrix T are exactly the eigenvectors for $\lambda = 1$.

If T has $\lambda = 1$ as an eigenvalue, no repeats, and $|\lambda_i| < 1$ for all other λ_i

$u^{(k)} \to u^*$, where u^* is a fixed point.

Formula $u^{(k)} = \underbrace{c_1 \lambda_1^k}_{c_1 \sim 1} v_1 + \cdots + \underbrace{c_n \lambda_n^k v_n}_{\to 0}$

$\lambda_1 \dots \lambda_n$ are eigenvalues

$v_1 \dots v_n$ are eigenvectors

$c_1 \dots c_n$ determined by $u^{(0)}$.

$$T^k = (S \Lambda S^{-1})^k = S \Lambda^k S^{-1}$$

$$\Lambda^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}$$

If $\lambda = 1$

$$u^{(k)} = c_1 \overbrace{\lambda_1^k}^{\lambda=1} v_1 + \underbrace{\cdots + c_n \lambda_n^k v_n}_{(\lambda_i < 1)}$$

$$= c_1 v_1$$

$$u^{(k)} \longrightarrow c_1 v_1 = u^* \quad \text{, fixed point}$$

Markov processes

LIS + probabilities

$u^{(0)}$ = probability vector

T = regular transition matrix

columns sum to 1

T^k has all nonzero entries for some power k.

$u^{(k)} \longrightarrow u^*$ - average probability

- Random walk on a graph
  - #2 review

- $\dim(\mathrm{coker}(A_{inc})) = \#$ of independent circuits

- $\#v - \#e = 1 - \#$ ind. circ.

$$\#v - \#e = \chi(G)$$

Euler characteristic

#7 Review

Find the Jordan decomposition

of

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 9 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\det(C - \lambda I) = 0$$

$$= \det \begin{pmatrix} 2-\lambda & -1 & 0 \\ 9 & -4-\lambda & -3 \\ 0 & 0 & -1-\lambda \end{pmatrix} = 0$$

$$\begin{matrix} + & - & + \end{matrix}$$

$$\cancel{0 \det \begin{pmatrix} -1 & 0 \\ -4-\lambda & -3 \end{pmatrix}}$$

$$\cancel{- 0 \det \begin{pmatrix} 2-\lambda & 0 \\ 9 & -3 \end{pmatrix}}$$

$$+ (-1-\lambda) \det \begin{pmatrix} 2-\lambda & -1 \\ 9 & -4-\lambda \end{pmatrix}$$

$$(-1-\lambda)\big((2-\lambda)(-4-\lambda) + 9\big) = 0$$

$$(-1-\lambda)(-8 + 2\lambda + \lambda^2 + 9) = 0 \qquad \lambda = -1$$

$$-(\lambda+1)(\lambda^2 + 2\lambda + 1) = -(\lambda+1)^3 = 0 \qquad \lambda = -1 \qquad \lambda = -1$$

The eigenvector $V_{-1} = \ker(C-(-1)I)$

$= \text{span}(1,3,0).$

$\lambda = -1$

$\lambda = -1$

$\lambda = -1$

We need 2 generalized eigenvectors

$v_1 = (1,3,0)$ The Jordan chain can be found by

$\underline{v_1, w_1, w_2}$

solving v_1, w_1, w_2

* $((C-(-1)I)\underline{w_1} = v_1$

* $((C-(-1)I)\underline{w_2} = w_1$

* $w_1 = \cancel{\frac{y}{3}\begin{pmatrix}1\\3\\0\end{pmatrix}} + \underbrace{\begin{pmatrix}1/3\\0\\0\end{pmatrix}}_{w_1}$ $\qquad w_1 = \begin{pmatrix}1/3\\0\\0\end{pmatrix}$

* $w_2 = \cancel{\frac{y}{3}\begin{pmatrix}1\\3\\0\end{pmatrix}} + \underline{\begin{pmatrix}1/9\\0\\1/3\end{pmatrix}}$ $\qquad w_2 = \begin{pmatrix}1/9\\0\\1/3\end{pmatrix}$

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 9 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix}$$

$$C = SJS^{-1}$$

$$S = \begin{array}{c} \begin{array}{ccc} v_1 & w_1 & w_2 \end{array} \\ \begin{pmatrix} 1 & 1/3 & 1/9 \\ 3 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \end{array}$$

$$J = \begin{array}{c} \begin{array}{ccc} \text{no } 1 & \text{yes } 1 & \text{yes } 1 \end{array} \\ \begin{pmatrix} -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{pmatrix} \end{array}$$

Schur Decomp.

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

$\lambda = 0$, $\lambda = -4$

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

unitary = complex version of orthogonal

$B = U \Delta U^T$ where U is orthog.

Δ is UΔ w/ eigenvals on diag.

For 2×2

$\lambda = 0 \quad \lambda = -4$

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

① Complete $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ to an orthonormal basis of $\mathbb{R}^2$

that forms U.

$$\Delta = U^T B U = \begin{pmatrix} 0 & ? \\ 0 & -4 \end{pmatrix}$$

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$(x, y) \perp (-y, x)$ in $\mathbb{R}^2$ |

$$\boxed{U = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}$$

$$\Delta = U^T B U = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 & -3 \\ 0 & -4 \end{pmatrix}}$$

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}_{\text{orth.}} \underbrace{\begin{pmatrix} 0 & -3 \\ 0 & -4 \end{pmatrix}}_{\text{upp }\Delta} \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Schur decomp. λ on the diagonal

HW #11 8.2.23

Show that AB BA have same eigenvalues.

① λ is an eigenvalue of AB and $\lambda \neq 0$

② $\lambda = 0$

Show: If λ is a eigenval for AB then it is for BA as well

$\Rightarrow$ If $ABv = \lambda v$ $v \neq 0$ then $\exists w \neq 0$ (there exists) s.t. $BAw = \lambda w$.

Case 1: If λ is an eigenval for AB, $\lambda \neq 0$

then $\underline{ABv = \lambda v}$, $v \neq 0$.

Claim: that $\underline{w = Bv}$ is an eigenvector of BA w/ eigenval $\lambda \neq 0$.

$$\underline{BAw} = BA(Bv) = B\overbrace{(AB)v} = B(\lambda v) = \lambda(Bv) = \underline{\lambda w}$$

w is eigenvec w/ λ value

Need $w \neq 0$

$Bv \neq 0$ since if $Bv = 0$ then $ABv = A \cdot 0 = 0$

$\lambda v = 0$.

$\lambda \neq 0$, $v \neq 0$. contradict.

$\underline{Bv \neq 0}$!

w is a nonzero eigenvector.

Case 1 done.

Case 2: $\lambda = 0$.

Want: If $ABv = 0$
then $BAw = 0$ for
some $w \neq 0$.

Case 2.1 $Bv = 0$

If $w = Bv = 0$
w can't be a
eigenvector

So $BA(Bv) = 0$ doesn't
tell us anything.

If A^{-1}, compute $\ker B$.

If A^{-1} doesn't exist, pick
$w \in \ker A$.

$BAw = 0$

Case 2.2 $Bv \neq 0 \Rightarrow w = Bv \neq 0$.

$BA(w) = BABv = 0$
$= 0w$. $\square$

#9 Review:

let B pos def matrix

w/ $B^2 = Q \Lambda Q^T$.

What is the spectral decomp of B in terms of Q, Λ?

- We need $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\lambda_1, \dots, \lambda_n$ eigenvals of B^2

- If λ_i is a eigenval for B^2

$\pm\sqrt{\lambda_i}$ is an eigenval for B.

- Pos def $\Rightarrow \lambda_i > 0$.

Claim:

- $B = Q \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} Q^T$

- In the first place

All eigenvalues of B are positive since B is pos def.

They should also square to $\lambda_1 \dots \lambda_n$.

- So the eigenvals of B are positive roots $\sqrt{\lambda_i}$.

Eigenvectors are same

$B^2 v = \lambda v \Leftrightarrow Bv = \sqrt{\lambda} v$

$$B = Q \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} Q^T$$

Q is the matrix of eigenvectors.

$$B^2 = Q \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \cancel{Q^T Q} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$$= Q \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} Q^T$$

$$= Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^T = Q \Lambda Q^T.$$

Exam 1

- Row reduction RREF

$A = LU$ L matrix of row operations

$A \to U$.

$r_j = c r_i + r_j$

$-c$ would go into (j, i) $i < j$.

swapping rows

$\downarrow$

$PA = LU$

Big Thm The following are equivalent for a square matrix A.

1. A^{-1} exists
2. $A \longrightarrow I$ by row reduction * used for computing A^{-1}.
3. $\ker(A) = 0$
4. Columns are independent (i.e. form a basis of $\mathbb{R}^n$)
5. Rows are independent *
6. A has n pivots ($\text{rank}(A) = n$)
7. $\det(A) \neq 0$. * fastest way to show A^{-1} exists or not.
8. $\lambda = 0$ is not an eigenvalue.

$$(A \mid I) \longrightarrow (I \mid A^{-1}).$$

$\text{rank}(A) = \text{rank}(A^T)$

$\dim \text{span}(\text{rows}) = \dim \text{span}(\text{columns})$

$= \#$ of leading 1's in RREF

4a Review

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}$$ these rows are the same!

(a) You should be able to just look at A and see one eigenvalue.

$\Rightarrow$ A does not have linearly independent rows.

$$r_2 - r_3 = 0$$

$\Rightarrow$ $\ker(A) \neq 0$

$\Rightarrow$ $\lambda = 0$ must an eigenvalue.

	L^1	L^2	L^∞
$\mathbb{R}^n$	$\sum \lvert v_i \rvert$	$\sqrt{\sum v_i^2}$	$\max\{\lvert v_i \rvert\}$
$C^0([a,b])$	$\int_a^b \lvert f(x) \rvert \, dx$	$\sqrt{\int_a^b f(x)^2 \, dx}$	$\max_{a \le x \le b}\{\lvert f(x) \rvert\}$ *

$\|v\| \qquad v = (v_1, \ldots, v_n)$ $\qquad \xi$

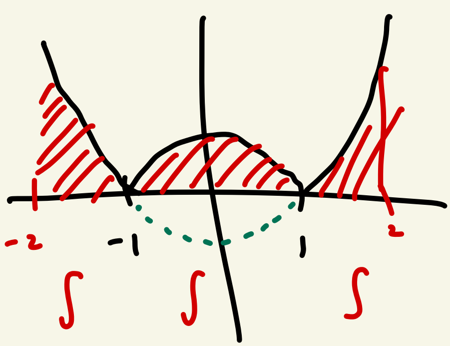
$f \in C^0$ $\qquad \Sigma \quad \xi$

Σ

Σ

Σ

$$\int_{-2}^{2} \lvert x^2 - 1 \rvert \, dx$$



$\xi \qquad xi$

$\xi \qquad \xi$

$\xi \qquad \xi$

Def Algebraic multiplicity of the eigenvalue is the number of λ repeats as a root of the characteristic polynomial.

Ex $(\lambda-2)^{3}(\lambda+1)^{2}=0$

$\lambda=2$ $\lambda=-1$

mult $=3$ mult $=2$

(geometric multiplicity)

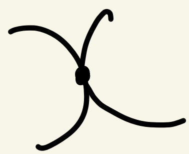
$\dim(V_\lambda) \leq$ alg mult of λ

$\dim(V_\lambda) =$ alg mult λ

$\Updownarrow$

for every repeat of λ, I have an eigenvector.

$$\frac{\sum h_i}{260} \cdot 300$$



$x^2+y^2=1$

$x^2-y^2+x-y=0$

$x = \pm$

$y = \pm$