Reminder:

Exam 2 is on Friday!

It will cover material through Wednesday.

Same format as exam 1.

Materials today or tomorrow.

Chapter 7.

§7.1 Linear Functions

Recall in Ch 2, we defined

vector spaces.

Vector Spaces

- $\mathbb{R}^n$
- polynomials
- $C^0[a,b]$, $C^\infty[a,b]$,

etc.

Same formulas, even though one inner product was an integral, or the dot product

But just as important as vector spaces, is functions between vector spaces.

<u>Def</u>: A linear function $T: V \longrightarrow W$, where V, W are vector spaces, is a function w/ inputs from V and outputs in W, such that

- $T(v_1 + v_2) = T(v_1) + T(v_2)$ }
- $T(c v_1) = c\,T(v_1)$, }

where $c \in \mathbb{R}$, $v_1\, v_2 \in V$.

Ex: let $V = \mathbb{R}^2$ $W = \mathbb{R}^2$

A linear function

$$T: \underline{\mathbb{R}^2} \longrightarrow \underline{\mathbb{R}^2}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \underline{f}(x,y) \\ \underline{g}(x,y) \end{pmatrix}$$

and $T(\vec{x} + \vec{v})$

$$= T(\vec{x}) + T(\vec{v})$$

$$T\begin{pmatrix} cx \\ cy \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \underline{2x+y} \\ \underline{-x+2y} \end{pmatrix}$$ is linear Pf later }

It's a linear function because these are lines!

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ \sin(xy) \end{pmatrix}$$ not linear!

Ex For any vector spaces V, W

$$T(v) = 0_W \in W. \quad \forall v.$$

This is a linear function $T: V \to W$.

Pf . $T(v_1 + v_2)$

$$= 0 = T(v_1) + T(v_2)$$

. $T(cv_1) = 0 = c \cdot 0 = c \cdot T(v_1)$

Ex: Let $V = \mathbb{R}^1$ $W = \mathbb{R}^1$.

$T: \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$T(x) = 2x$ is linear. $y = 2x$

Pf $T(x+y) = 2(x+y)$

$$= 2x + 2y = T(x) + T(y)$$

$$T(cx) = 2(cx) = c(2x) = cT(x).$$

Prop Any linear function $T: \mathbb{R} \to \mathbb{R}$ is of the form $T(x) = \underline{a}x$, for some $a \in \mathbb{R}$. $\rightsquigarrow$ "slope"

Pf: Let $T: \mathbb{R} \to \mathbb{R}$ be a linear function.

We know that $T(x+y) = T(x) + T(y)$

$T(cx) = cT(x)$.

Note that $x = x \cdot 1$, we can treat x as a scalar, 1 as a vector.

$$T(x) = T(x \cdot 1) = xT(1)$$

Let $T(1) = a$, so that

$T(x) = ax$. $\square$

Note: $T(x) = ax + b$ is **not** linear if $b \neq 0$!! We only want lines through the origin!

Ex $T(x) = x + 1$ not linear

$$T(x+y) = x + y + 1$$

$\neq$

$$T(x) + T(y) = (x+1) + (y+1)$$
$$= x + y + 2$$

Prop: If $T : V \to W$ is a linear function, then $T(0_V) = 0_W$.

Pf $T(\vec{0}_V) = T(0 \cdot \vec{0}_V)$

$$= 0 \, T(\vec{0}_V) = \vec{0}_W$$

Let's think about linear functions

$$T : \mathbb{R}^{\overline{n}} \longrightarrow \mathbb{R}^{\overline{m}}.$$

flipped

<u>Ex</u> Let A be any $\overline{m} \times \overline{n}$ matrix.

Then $T(\vec{x}) = A\vec{x}$ is a linear function from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

"A is a slope".

<u>Pf</u>: $T(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y})$

$$= A\vec{x} + A\vec{y} = T(x) + T(y)$$

We already knew matrix mult is distributive, and you can pull out scalars.

$$T(c\vec{x}) = A(c\vec{x}) = c(A\vec{x})$$

$$= c\,T(\vec{x})$$

Thm Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Then

$$T(x) = Ax \quad \text{for some}$$

matrix $A \in M_{m \times n}(\mathbb{R})$.

Pf: Last time $T(1)$ was the slope.

This time... $A = \left(T(e_1) \; \dots \; T(e_n) \right)$.

Let $e_1, \dots, e_n$ be standard basis of $\mathbb{R}^n$.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ etc.}$$

$\hat{e}_1, \dots, \hat{e}_m$ be the standard basis of $\mathbb{R}^m$.

let $T(e_i) = \vec{a}_i \in \mathbb{R}^m$.

$$= \underline{a_{1i}}\, \hat{e}_1 + \dots + \underline{a_{mi}}\, \hat{e}_m$$

Then for any $v \in \mathbb{R}^n$. $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$.

$$\underline{T(\vec{v})} = T(v_1 e_1 + \dots + v_n e_n)$$

$$= T(v_1 e_1) + \dots + T(v_n e_n)$$

$$= v_1 T(e_1) + \dots + v_n T(e_n)$$

$$= v_1 a_1 + \dots + v_n a_n$$

If we let $A = (a_1 \dots a_n) \in M_{m \times n}$

then $v_1 \vec{a}_1 + \dots + v_n \vec{a}_n$

$$= \underline{Av}$$

So $T(v) = Av$. $\square$

Ex $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ -x + 2y \end{pmatrix}.$$ It's linear!

$$T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 2(x_1 + x_2) + (y_1 + y_2) \\ -(x_1 + x_2) + 2(y_1 + y_2) \end{pmatrix}$$

$$= \begin{pmatrix} (2x_1 + y_1) + (2x_2 + y_2) \\ (-x_1 + 2y_1) + (-x_2 + 2y_2) \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + y_1 \\ -x_1 + 2y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 + y_2 \\ -x_2 + 2y_2 \end{pmatrix}$$

$$= T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

$T(cx) = cT(x)$ is similar.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -x+2y \end{pmatrix}$

There's a corresponding matrix A such that $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.

A is 2×2.

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 0 \\ -1 + 2\cdot 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 1 \\ -0 + 2\cdot 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \underline{-1} & \underline{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ as desired.

<u>Ex</u>: let $V = C^0[a,b]$

$W = \mathbb{R}$

let $T : V \longrightarrow W$

$= T : C^0[a,b] \longrightarrow \mathbb{R}$

functions are inputs!

$$T(f) = \int_a^b f(x)\,dx \in \mathbb{R}.$$

This is a linear function!

<u>Pf</u> $T(f+g)$

$$= \int_a^b f(x) + g(x)\,dx$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$= T(f) + T(g).$$

$$T(cf) = \int_a^b c f(x)\, dx$$

$$= c \int_a^b f(x)\, dx$$

$$= c\, T(f). \qquad \square$$

Ex let $V = C^1[a,b]$

= continuously differentiable functions

$W = C^0[a,b]$

$\frac{d}{dx} : \underline{C^1[a,b]} \longrightarrow \underline{C^0[a,b]}$

is a linear function!

$\frac{d}{dx}(f) = f'.$

input

output

Pf $$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

$$\frac{d}{dx}(cf) = c\frac{d}{dx}(f)$$ $\square$

So studying linear functions $T: V \to W$ in general is like studying matrices, derivatives, integrals, at once.