Yesterday...

We defined a linear function

$$T : V \longrightarrow W$$

$$T(v+w) = T(v) + T(w)$$

$$T(cv) = cT(v)$$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \vec{x} \longmapsto A\vec{x}$$

$A$ is $m \times n$

$$\frac{d}{dx} : C^1[a,b] \longrightarrow C^0[a,b]$$

$$\int \_\, dx : C^0[a,b] \longrightarrow \mathbb{R}$$

**Def**: Let $V$ and $W$ be (real) vector spaces.

Define $\mathrm{Hom}(V,W)$ to be the set of all linear functions from $V$ to $W$.

**Aside**: "Hom" is short for homomorphism = function w/ properties

**Prop**: $\mathrm{Hom}(V,W)$ is a real vector space also!

**Pf**: In theory we need define addition and scalar mult

- $T_1 + T_2$. · $cT_1$.

+ 7 axioms

let $T_1 : V \to W$ $T_2 : V \to W$

$T_1, T_2 \in \text{Hom}(V, W)$.

Define

$$(T_1 + T_2)(v) = T_1(v) + T_2(v).$$

Claim: $T_1 + T_2$ is a linear function also!

$$\begin{aligned}
(T_1 + T_2)(u+v) &= T_1(u+v) + T_2(u+v) \\
&= \underbrace{T_1(u) + T_1(v)}_{T_1 \text{ linear}} + \underbrace{T_2(u) + T_2(v)}_{T_2 \text{ is linear}} \\
&= T_1(u) + T_2(u) + T_1(v) + T_2(v) \\
&= (T_1 + T_2)(u) + (T_1 + T_2)(v)
\end{aligned}$$

Define $(cT)(v) = cT(v)$, $c \in \mathbb{R}$

$cT$ is also linear.

Turns out that these $T_1 + T_2$ and $cT$ satisfy vector space axioms, so

$\mathrm{Hom}(V, W)$ is a vector space!

---

<u>Ex</u> All linear functions from $\mathbb{R}^1 \longrightarrow \mathbb{R}^1$ are of the form $T(x) = ax$.

$$\mathrm{Hom}(\mathbb{R}^1, \mathbb{R}^1) = \left\{ \begin{array}{l} \text{all linear functions} \\ \text{from } \mathbb{R}^1 \longrightarrow \mathbb{R}^1 \end{array} \right\}$$

$$= \left\{ \text{all } \begin{array}{l} \text{functions of the form} \\ T(x) = ax \mid a \in \mathbb{R} \end{array} \right\}$$

slope is all that matters

$$ax + bx = (a+b)x$$

$$T_a + T_b = T_{a+b}$$

$$= \{a \mid a \in \mathbb{R}\} = \mathbb{R}$$

Ex All linear functions from $\mathbb{R}^n \to \mathbb{R}^m$

are of the form $T(x) = Ax$

$A$ is an $m \times n$ matrix.

$\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

$= \{\text{all linear functions } \mathbb{R}^n \to \mathbb{R}^m\}$

$= \{\text{all functions } T(x) = \underline{A}x\}$

only relevant piece of info

$= \{A \mid A \in M_{m\times n}(\mathbb{R})\}$

$= M_{m\times n}(\mathbb{R})$

Remember $M_{m\times n}(\mathbb{R})$ is a vector space!

$A + B$, $cA$  $\dim(M_{m\times n}(\mathbb{R})) = mn$.

So $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is a v.s too.

**Def**: Let $V$ be a vector space.

Define $V^*$, "$V$ dual", by

$$V^* = \mathrm{Hom}(V, \mathbb{R}^1)$$

$$= \{\text{all linear functions } V \to \mathbb{R}\}$$

**Ex** $(\mathbb{R}^n)^*$

$= \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$

$=$ all linear functions from $\mathbb{R}^n \to \mathbb{R}^1$

$=$ all $1 \times n$ matrices $(m=1)$

$=$ row vectors $(a_1 \ a_2 \ \dots \ a_n)$

If $\mathbb{R}^n$ is all column vectors, $(\mathbb{R}^n)^*$ all row vectors.

Def: Given a linear function

$T: V \to W$, we can

define a dual function

$T^*: W^* \to V^*$

reverses order!

$T^*$

input: a linear function $= W^*$

$W \to \mathbb{R}$

output: a linear function

$V \to \mathbb{R} = V^*$

$T^*$ changes domain of a function

$T^*(f)$ is a function $V \to \mathbb{R}$ if

$f: W \to \mathbb{R}$.

What's actual formula for $T^*(f)$?

Given input $f: W \to \mathbb{R}$, output of $T^*$ is

$f(T): V \to \mathbb{R}$

$V \xrightarrow{T} W \xrightarrow{f} \mathbb{R}$

$T : V \to W \qquad f : W \to \mathbb{R}$ linear

$$\underline{T^*}(f) = \underline{f \circ T = f(T)}$$

Remember Chain Rule

$$\frac{d}{dx}\left(\boxed{g(f(x))}\right) = \frac{dg}{df} \cdot \frac{df}{dx}$$

$g(f(x))$ is the same as evaluating $f$ and then $g$

$$\left.\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \qquad g : \mathbb{R} \to \mathbb{R} \\ g(f) : \mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R} \end{array}\right\}$$

Now

$$\boxed{V} \xrightarrow{T} W \xrightarrow[f \in W^*]{} \boxed{\mathbb{R}}$$

$$T^*(f) = f(T) : V \to \mathbb{R} \ \underline{\text{linear}}\,!$$

Ex

let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$   $A$ is $m \times n$

$T(x) = Ax$

$T^* : (\mathbb{R}^m)^* \longrightarrow (\mathbb{R}^n)^*$

$\qquad\quad \|$ $\qquad\qquad\qquad\qquad\qquad \|$

$T^*$ : $m$ row vector $\longrightarrow$ $n$ row vectors

$$T^*((a_1 \ldots a_m)) = (b_1 \ldots b_n)$$

$$(1 \times m) \times (m \times n) = 1 \times n$$

Turns out..

$$T^*(x_1 \ldots x_m) = (x_1 \ldots x_m) A$$

$$\rightsquigarrow \quad T^* \mapsto A^T, \; A^T \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

Pf :

$$T^* : (\mathbb{R}^m)^* \longrightarrow (\mathbb{R}^n)^*$$

$\shortparallel$

all row vectors

$$(a_1 \dots a_m) : \mathbb{R}^m \longrightarrow \mathbb{R}$$

output

$$T^*(a_1 \dots a_m)$$

$$\left( = \quad \mathbb{R}^n \xrightarrow[A]{} \mathbb{R}^m \xrightarrow[(a_1 \dots a_m)]{} \mathbb{R} \right)$$

$$= (a_1 \dots a_m) \circ A$$

$$= (a_1 \dots a_m) A$$

rows, columns, matrices are specific to $\mathbb{R}^n$.

But no rows, columns, or matrices for vector spaces like $C^0[a,b]$

If a function $f(x)$ is a "column vector", a "row vector" would a linear function

$$\underbrace{C^0[a,b] \longrightarrow \mathbb{R}}_{}$$

"row vector"

$$T_f(g) = \int_a^b f(x)g(x)\,dx$$

$$= \langle f, g \rangle \in C^0[a,b]^*$$

OR more generally

"columns" "row"

$$f \longrightarrow \langle f, - \rangle : V \to \mathbb{R}$$

---

Let $V$ be a f.d real inner product space.

Then all linear functions from $V \to \mathbb{R}$ are of the form

$$T(v) = \langle a, v \rangle \quad \text{for some } a \in V.$$

$$V^* = \{\text{all linear functions } V \to \mathbb{R}\}$$

$$= \{\text{all } T(v) = \langle a, v \rangle\}$$

$$= \{\langle a, - \rangle \mid a \in V\}$$

= "row vectors of $V$"

(This is a bijection between $V \to V^*$.)

## § 7.2 Transformations and Change of Basis

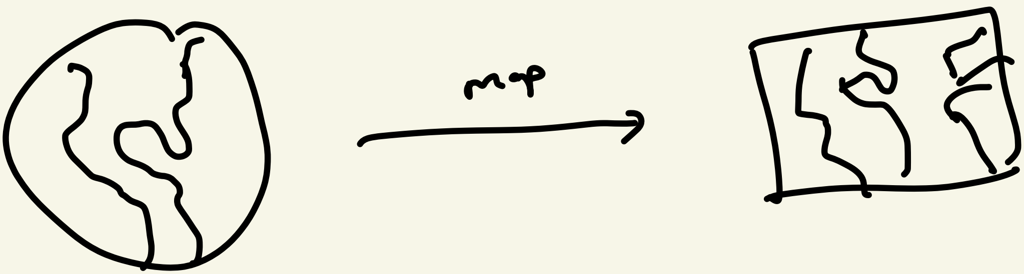
In 7.1 I tried to use

linear function.

Linear function is the same as

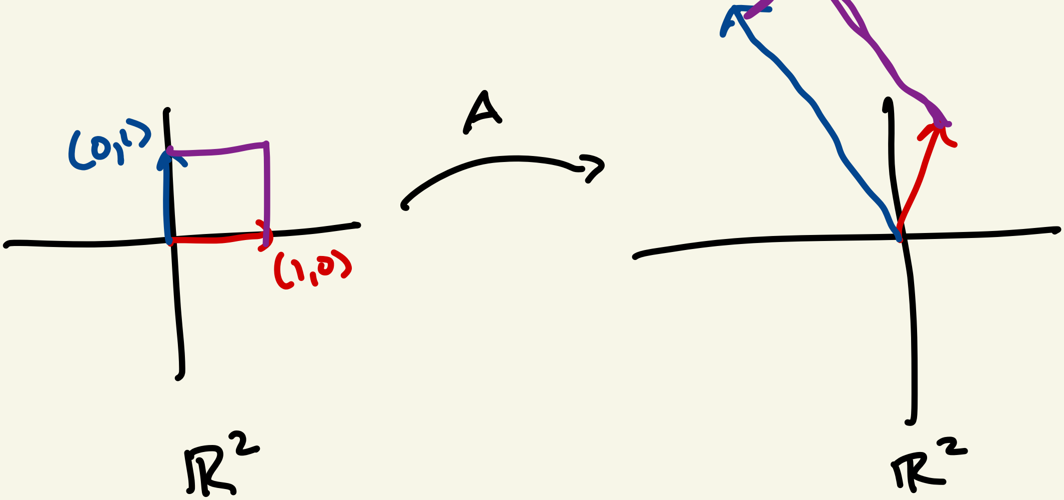
a linear transformation,

and a linear map.




$f : A \longrightarrow B$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ are just 2x2 matrices




$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

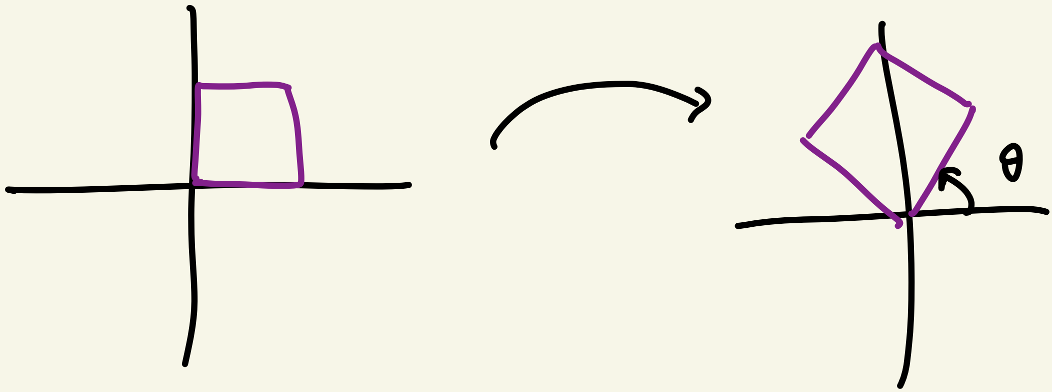
The word "transformation" refers to this picture.

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

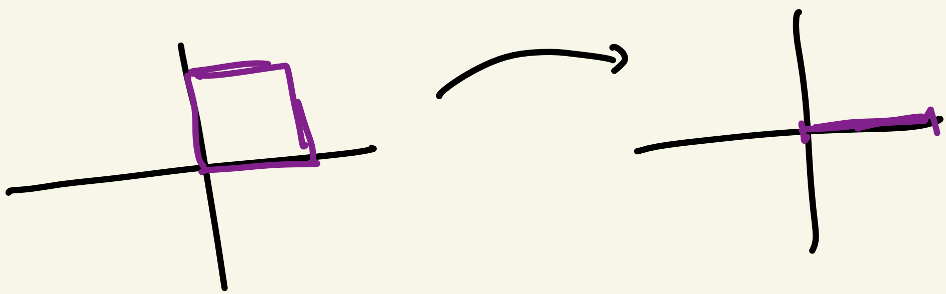
$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

So $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ stretches out and rotates the box somehow.

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$




$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



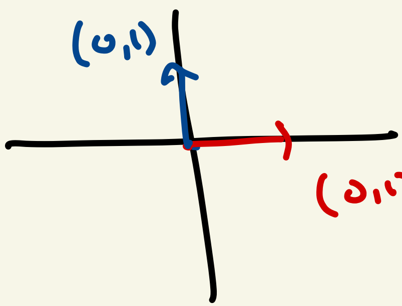
$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

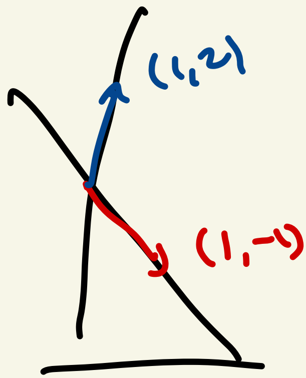
<u>Projects</u> $\begin{pmatrix} x \\ y \end{pmatrix}$ onto the $x$-axis.

Something unfortunate! There isn't always a best basis.

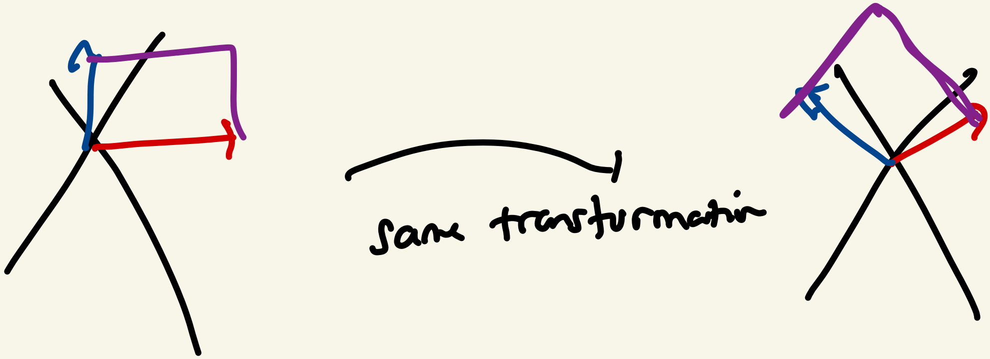
$\mathbb{R}^n$ $\quad \{e_1, \dots, e_n\}$ is usually a good basis

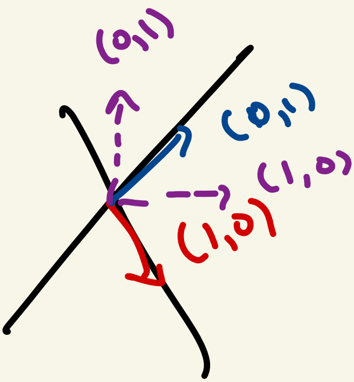
$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, $\quad e_1, e_2$ isn't the best basis




Idea: If draw axes differently, the transformation is the same, but matrix will be different.

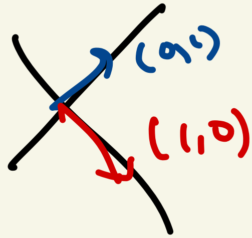



Since the axes are different, the matrix mult will be different.

$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ is <u>not</u> the right matrix in these weird axes!

Given another choice of axes, what is the matrix?




In standard coordinates

$$\begin{pmatrix}a\\b\end{pmatrix} = a\underline{\begin{pmatrix}1\\0\end{pmatrix}} + b\underline{\begin{pmatrix}0\\1\end{pmatrix}}$$

$$= a\underline{e_1} + b\underline{e_2}$$

In $\mathbb{R}^2$ w/ basis $v_1, v_2$,

$\begin{pmatrix}a\\b\end{pmatrix}$ might refer to

$av_1 + bv_2$.

<u>Ex</u> If $v_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ $v_2 = \begin{pmatrix}1\\-2\end{pmatrix}$

then in $v_1 v_2$ coordinates

$$\underline{\begin{pmatrix}1\\0\end{pmatrix}_{v_1,v_2}} = 1v_1 + 0v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix}_{v_1,v_2} = 0v_1 + 1v_2 = \begin{pmatrix}1\\-2\end{pmatrix}$$

Q: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

and in standard coordinates

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

If $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \end{pmatrix}_{v_1 v_2} = B\begin{pmatrix} x \\ y \end{pmatrix}_{v_1 v_2}$$

How do we calculate $B$ in terms of $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$?

$$\begin{pmatrix} x \\ y \end{pmatrix}_{v_1 v_2} = x\vec{v}_1 + y\vec{v}_2$$

$$= \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_{v_1 v_2} = S\begin{pmatrix} x \\ y \end{pmatrix}. \quad S = (v_1\ v_2)$$

So $S$ is a matrix which converts from $e_1 e_2$ coord. to $v_1 v_2$ coord

Ex Write $\begin{pmatrix}5\\3\end{pmatrix}$ in $v_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ $v_2 = \begin{pmatrix}1\\-2\end{pmatrix}$

coordinates

$$S = \begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}$$

$$\begin{pmatrix}5\\3\end{pmatrix}_{v_1v_2} = \begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}\begin{pmatrix}5\\3\end{pmatrix}$$

$$= \begin{pmatrix}8\\-11\end{pmatrix}$$

$$\underline{\begin{pmatrix}5\\3\end{pmatrix}}_{v_1v_2} = 5v_1 + 3v_2 = \begin{pmatrix}8\\-11\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}_{v_1v_2} = \begin{pmatrix}5\\3\end{pmatrix}_{e_1e_2}$$

$$x\begin{pmatrix}1\\-1\end{pmatrix} + y\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}5\\3\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} = \underline{\begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}^{-1}}\begin{pmatrix}5\\3\end{pmatrix}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\underset{\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}}{\mathbb{R}^2} \arrow[r, "B=??"'] \arrow[d, shift left, "S"] & \underset{\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}}{\mathbb{R}^2} \arrow[d, shift left, "S"] \\
\underset{e_1 e_2}{\mathbb{R}^2} \arrow[u, shift left, "S^{-1}"] \arrow[r, "\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}=A"'] & \underset{e_1 e_2}{\mathbb{R}^2} \arrow[u, shift left, "S^{-1}"]
\end{tikzcd}
$$

$$
B = \underbrace{\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}^{-1}}_{S^{-1}} \underbrace{\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}}_{S}
$$

$$
B = S^{-1} A S
$$

* * $\boxed{B = S^{-1}AS}$ * * !!

A is a given

$v_1\, v_2$ also a given

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

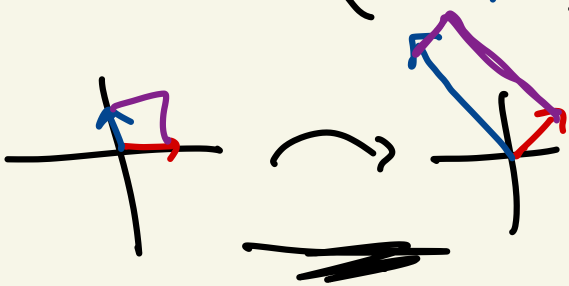
$$= \begin{pmatrix} \boxed{2} & 0 \\ 0 & \boxed{3} \end{pmatrix}$$

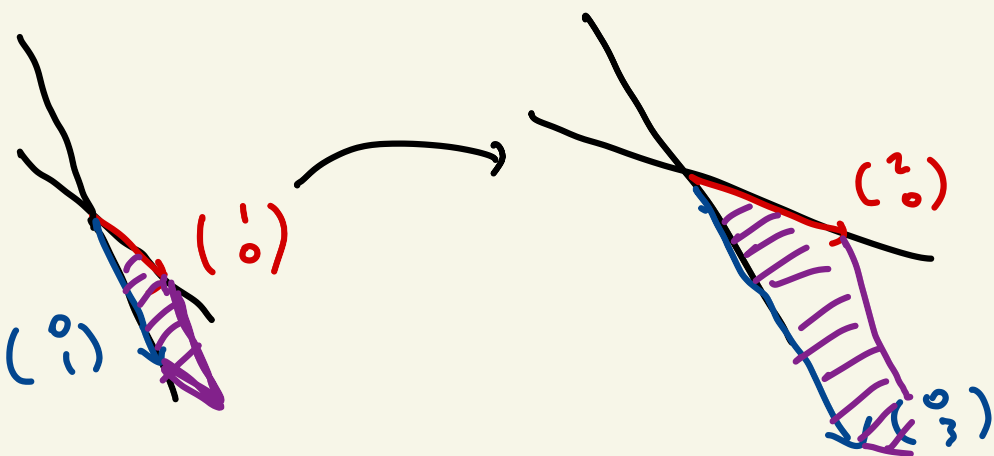
This the matrix for the transformation

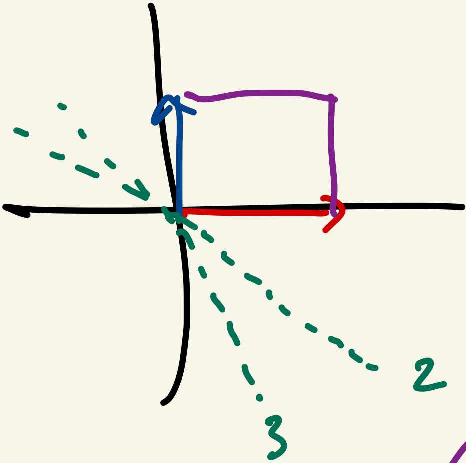
i.e. $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *

$v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ *

coordinates

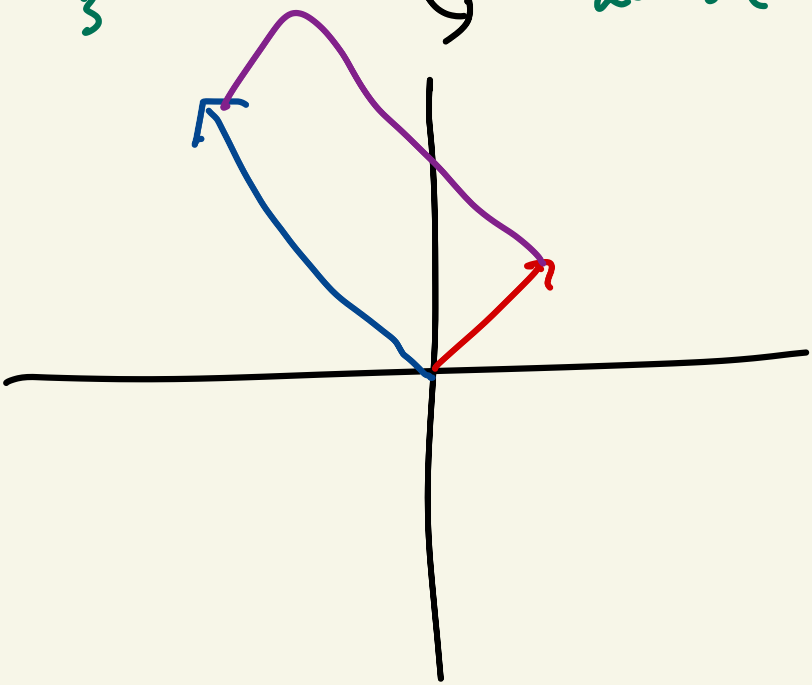







stretch by 2 along $\begin{pmatrix}1\\-1\end{pmatrix}$

stretch by 3 along $\begin{pmatrix}1\\-2\end{pmatrix}$



# Change of Basis in general

Given a matrix $A \in M_{n \times n}(\mathbb{R})$

$$A : \underset{e_1 \ldots e_n}{\mathbb{R}^n} \longrightarrow \underset{e_1 \ldots e_n}{\mathbb{R}^n}$$

Let $v_1 \ldots v_n$ be a basis of $\mathbb{R}^n$.

Let $S = (v_1 \ldots v_n)$, $S^{-1}$ exists

If $B : \underset{v_1 \ldots v_n}{\mathbb{R}^n} \longrightarrow \underset{v_1 \ldots v_n}{\mathbb{R}^n}$

is the same transformation,

then $B = S^{-1} A S$.

Pf:

$$
\begin{tikzcd}[row sep=large, column sep=huge]
\underset{v_1 \cdots v_n}{\mathbb{R}^n} \arrow[r, "B"] \arrow[d, shift left, "S"] & \underset{v_1 \cdots v_n}{\mathbb{R}^n} \arrow[d, shift left, "S"] \\
\underset{e_1 \cdots e_n}{\mathbb{R}^n} \arrow[r, "A"'] \arrow[u, shift left, "S^{-1}"] & \underset{e_1 \cdots e_n}{\mathbb{R}^n} \arrow[u, shift left, "S^{-1}"]
\end{tikzcd}
$$

This the first of many commutative diagrams in algebra.

$$(S\vec{x}) = x_1 v_1 + \dots + x_n v_n$$

$$\text{regular} = \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{v_i}}$$

$B : \underset{v_1 \cdots v_n}{\mathbb{R}^n} \longrightarrow \underset{v_1 \cdots v_n}{\mathbb{R}^n}$ being the same transformation just says that this box "commutes".

$$\boxed{B = S^{-1}AS}\,! \qquad S = (v_1 \cdots v_n).$$

How do we convert $A$ from

$w_1 \dots w_n$ coordinates to

$v_1 \dots v_n$ coordinates?

$T = (w_1 \dots w_n)$ $S = (v_1 \dots v_n)$

$$
\begin{tikzcd}
\mathbb{R}^n_{v_i} \arrow[r, "B"] \arrow[d, shift right, "S"'] & \mathbb{R}^n_{v_i} \arrow[d, shift right, "S"'] \\
\mathbb{R}^n_{e_i} \arrow[r, "A'"] \arrow[u, shift right, "S^{-1}"'] \arrow[d, shift right, "T^{-1}"'] & \mathbb{R}^n_{e_i} \arrow[u, shift right, "S^{-1}"'] \arrow[d, shift right, "T^{-1}"'] \\
\mathbb{R}^n_{w_i} \arrow[r, "A"'] \arrow[u, shift right, "T"'] & \mathbb{R}^n_{w_i} \arrow[u, shift right, "T"']
\end{tikzcd}
$$

$T^{-1}S$ is the vectors $v_i$ in terms of the $w_j$.

The columns of $T^{-1}S$ convert from $v_i$ to $w_i$.

$$B = S^{-1}TAT^{-1}S = (T^{-1}S)^{-1}A\underbrace{(T^{-1}S)}$$