§7.4 Linear systems

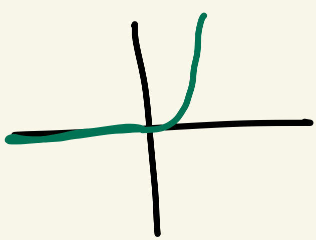
As we've shown,

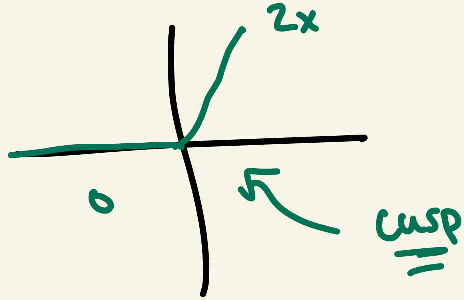
$$\frac{d}{dx} : C^1([a,b]) \longrightarrow C^0[a,b]$$

C^1 - vectors are differentiable functions on $[a,b]$ such that f' is cts

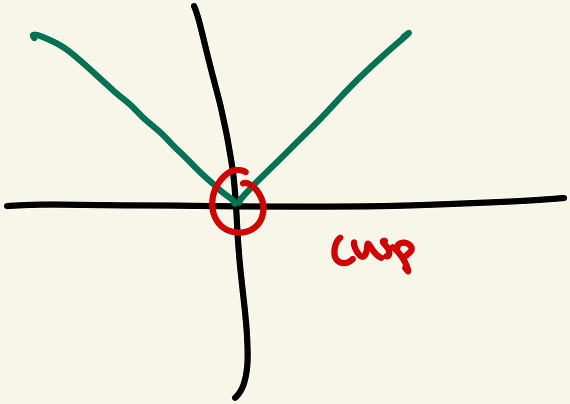
$f(x) = \cos x, \sin x, e^x,$ polynomials, etc ...

and $f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$, $f(x) \in C^1$





$f(x) = |x|$, $f \notin C^1([-1,1])$



$C^0([a,b])$ is just cts functions

$C^1[a,b]$ is a function w/ at least one cts derivative

$C^2[a,b]$ is the vector space of functions such that f'' is cts

$C^3[a,b]$ is the vector space of functions such that f''' is cts.

$\vdots$

Def $C^n[a,b]$ is the vector space of functions such that $f^{(n)}$ exists and is continuous.

Def / Prop

$$\frac{d^n}{dx^n} : \underline{C^n[a,b]} \longrightarrow C^0[a,b]$$

is a linear transformation.

Note! $\frac{d^n}{dx^n}$ ~~is~~ cannot have domain $C^0[a,b]$

$|x| \in C^0$ but $\frac{d}{dx}|x|$

$$= \begin{cases} -1 & x < 0 \\ \text{undef} & x = 0 \\ 1 & x > 0 \end{cases}$$

not a function on $\mathbb{R}$

Def A differential operator is a linear function

$$C^n(X) \longrightarrow C^0(X)$$

of the form

$$D = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_1(x)\frac{d}{dx} + a_0(x)\frac{d^0}{dx^0}$$

where $X = [a,b]$, (a,b) or $\mathbb{R}$.

$$D(f) = a_n(x)\frac{d^n f}{dx^n} + \dots + a_1(x)\frac{df}{dx} + a_0(x)f(x)$$

Since D is a linear combination of $a_i(x)\frac{d^i}{dx^i}$, each of which is linear, then D is linear.

A linear ordinary differential equation

is of the form

$$D(u) = f \}$$

for $u \in C^n[a,b]$, D a

linear operator.

Ex: $D = 1\dfrac{d^2}{dx} + 1\dfrac{d^0}{dx^0}$

$$\frac{d^2u}{dx^2} + u = 3x - 2\cos x$$

$$u'' + u = \underbrace{3x - 2\cos x}_{f}$$

Ex: $A : \mathbb{R}^n \to \mathbb{R}^m$

$$A\vec{x} = \vec{b} \qquad D(u) = f$$

Both examples of linear systems in general.

Def: Let $T: U \to V$ be a linear transformation.

Then a linear system is an equation of the form $T(u) = v$ for some $u \in U$, $v \in V$.

Ex $A: \mathbb{R}^n \to \mathbb{R}^m$

$Ax = b$ is a linear system

$D: C^n[a,b] \to C^0[a,b]$

$D(u) = f$ (l. ord. diff eq).

$f: A \to B$, $f^{-1}(b) = \{a \in A \mid f(a) = b\}$

Solving a linear system $T(u) = v$, is the same as finding the preimage of v under T.

Def: let $T: U \to V$.

Define the Kernel of T to be

$$\ker(T) = \{ u \in U \mid T(u) = 0 \}.$$

Ex A as a matrix, $A: \mathbb{R}^n \to \mathbb{R}^m$

$$\ker(A) = \{ x \mid Ax = 0 \}$$

$$= \ker(A) \leftarrow \text{as a transformation}$$

Ex $D = \frac{d^2}{dx^2} + \frac{d^0}{dx^0}$ $D(u) = u'' + u$

The kernel of D is the set of functions in $C^2[a,b]$

s.t. $D(u) = 0$

$$u'' + u = 0.$$

(Recall, this is a homog. eq.)

The first step in solving $T(u)=v$,
is to usually to solve $T(u)=0$,
i.e. find the kernel of T.

<u>Superposition Principle</u> and <u>7.38</u>

Let $T: U \to V$. $T(u)=v$
a linear system.

Superpos. principle

① $\ker(T)$ is a subspace
(if $z_1, \dots z_n \in \ker(T)$
then so is $c_1 z_1 + \dots + c_k z_n$.)

② $T(u)=v$ has a solution
iff $v \in \operatorname{img}(T)$
(Really just a definition
of $\operatorname{img}(T)$)

$$\operatorname{Img}(T) = \{ v \in V \mid T(u)=v \text{ for some } u \}$$

3 Solutions to $T(u) = v$

are of the form

$$u = u^* + z,$$

where u^* is one particular solution

and $z \in \ker(T)$.

Ex Calc III

$u'' + u = x$

1

find one solution

Guessing until you find one

$u'' + u = 0$

finding the kernel of $D = \frac{d^2}{dx^2} + \frac{d^0}{dx^0}$

$$u = u^* \qquad + \qquad z$$

Matrix systems have the same property!

$$\begin{array}{cccc} & & * \text{ free} & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \longleftarrow (A \mid \vec{b})$$

$$x + z = 2$$
$$y + z = 3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - z \\ 3 - z \\ z \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\text{kernel of } A} z$$

$$u = u^* + z$$

Thm Let $T(u) = c_1 \underline{f_1} + \dots + c_k \underline{f_k}$.

Let u_i^* be a particular solution to $T(u) = f_i$.

Then the general solution has the form

$$u = c_1 u_1^* + \dots + c_k u_k^* + z$$

where $z \in \ker(T)$.

Pf $u = c_1 u_1^* + \dots + c_k u_k^* + z$ is a solution since

$$\begin{aligned} T(u) &= T(c_1 u^* + \dots + c_k u_k^* + z) \\ &= c_1 T(u_1^*) + \dots + c_k T(u_k^*) + \cancelto{0}{T(z)} \\ &= c_1 f_1 + \dots + c_k f_k \end{aligned}$$

If u is the general solution

$$u - c_1u_1^* - \dots - c_ku_k^* \in \ker(T)$$

$$T(u - c_1u_1^* - \dots - c_ku_k^*)$$

$$= T(u) - c_1T(u_1^*) - c_2T(u_2^*) - \dots - c_kT(u_k^*)$$

$$= T(u) - c_1f_1 - \dots - c_kf_k$$

$$= c_1f_1 + \dots + c_kf_k - c_1f_1 - \dots - c_kf_k$$

$$= 0.$$

$$u - c_1u^* - \dots - c_ku_k^* = z \in \ker(T)$$

$$u = c_1u_1^* + \dots + c_ku_k^* + z$$

$\square$

Ex :

$D : C^2[a,b] \longrightarrow C^0[a,b]$

$D = \frac{d^2}{dx^2} + \frac{d^0}{dx^0}$, $D(u) = u'' + u$

$D(u) = 3x - 2\cos x$, $u(x)$

$u'' + u = 3x - 2\cos x$

① $u'' + u = x$ particular sol

$u_1^* = x$

② $u'' + u = \cos x$

$u = a\cos x + b\sin x$ ✗

$u = ax\cos x + b\,x\sin x$ $a = 0$, $b = \frac{1}{2}$

$u_2^* = \frac{1}{2}x\sin x$

③ $u'' + u = 0$

$u'' + u = 0$ *

-$a_n(x) \neq 0$
on $a \le x \le b$
· a_i cts

<u>Thm</u> Let D be a <u>nonsingular</u> differential operator

$$C^n[a,b] \to C^0[a,b]$$

$$D = \underbrace{a_n(x)\frac{d^n}{dx^n}} + \cdots + a_1(x)\frac{d^1}{dx} + a_0(x)$$

Then $\ker(D)$ is n-dim.

$u'' + u = 0$ has 2 independent solutions.

$u = e^{rx}$

$$r^2 e^{rx} + e^{rx} = 0$$

$$(r^2+1)e^{rx} = 0$$

$$r^2 + 1 = 0$$

$r = \pm i$

$u = e^{ix}$

$u = e^{-ix}$

$$\left.\begin{aligned} e^{ix} &= \underline{\cos x} + i\,\underline{\sin x} \\ e^{-ix} &= \underline{\cos x} - i\,\underline{\sin x} \end{aligned}\right\} \longrightarrow \text{interesting example of a general principle}$$

$u = a\cos x + b\sin x$

is the kernel of D.

Every solution to $u'' + u = 3x - 2\cos x$

$$u = a\cos x + b\sin x + 3u_1^* - 2u_2^*$$

$$= \underline{a\cos x + b\sin x + 3x + x\sin x}$$

$$u'' = -u$$