Reminder: Exam Tomorrow 7/10

Last time:

$$u'' + u = 0$$

computing the kernel of the linear operator

$$D = \frac{d^2}{dx^2} + \frac{d^0}{dx^0}\ !$$

$$\left(= \frac{d^2}{dx^2} + 1 \right)$$

$$D(u) = u'' + u$$

Guess: $u = e^{rx}$, $(r^2+1)e^{rx} = 0$

$\rightsquigarrow$ $r = \pm i$

$u'' + u = 0$ is a diff eq
on the reals

$\underline{u = e^{ix}}$, $\underline{u = e^{-ix}}$

$u = \overline{\cos x} + i\,\overline{\sin x}$
$u = \underline{\cos x} - i\,\underline{\sin x}$

$u = a\cos x + b\sin x$

$\cos x, \sin x$ span the kernel of D.

This is a particular case of a
general principle.

Let V be a complex vector space.
(scalars = complex numbers).

Def We say V is conjugated if there exists a conjugation operation $\overline{v}$ s.t.

(a) $\overline{\overline{v}} = v \quad \forall v \in V$

(b) $\overline{u+v} = \overline{u} + \overline{v} \quad \forall u, v \in V$

(c) $\overline{\lambda v} = \overline{\lambda}\,\overline{v} \quad \forall \lambda \in \mathbb{C},\ v \in V.$

Ex: Conjugation on $\mathbb{C}^n$.

$$z = (z_1, \ldots, z_n)$$

$$\overline{z} = (\overline{z_1}, \ldots, \overline{z_n}).$$

Eg $\overline{(i,\ 1-i,\ 2+2i)} = (-i,\ 1+i,\ 2-2i)$

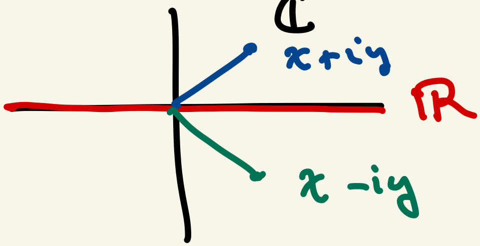
<u>Ex</u> let $C^{0}_{\mathbb{C}}[a,b]$ be complex valued functions

$$f : [a,b] \longrightarrow \mathbb{C}$$

- $f(x) = r(x) + i\,s(x)$

$$\overline{f(x)} = r(x) - i\,s(x)$$

So $C^{0}_{\mathbb{C}}[a,b]$ is a conjugated vector space.



For $z \in \mathbb{C}$, z is actually a real number if $\bar{z} = z$.

$$x + iy = x - iy$$

$$iy = -iy$$

$$y = -y$$

$$2y = 0$$

$$y = 0 \qquad \text{So } z = x \in \mathbb{R}.$$

Prop Let V be a conjugated complex vector space.

Then every vector $\vec{u} = \vec{v} + i\vec{w}$ where $\bar{v} = v$ and $\bar{w} = w$. $\quad u, v, w \in V$.

Pf: Idea − $u = v + iw$

then $v = \mathrm{Re}(u)$

$w = \mathrm{Im}(u)$

For normal complex numbers

$$z + \bar{z} = x + i\cancel{y} + x - i\cancel{y}$$

$$= 2x$$

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} = x$$

similarly. $\mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i} = y$

In V, let $v = \dfrac{u + \bar{u}}{2}$

$$w = \frac{u - \bar{u}}{2i}$$

$$\bar{v} = \overline{\left(\frac{u + \bar{u}}{2}\right)} = \frac{1}{2}\left(\frac{\bar{u} + \bar{\bar{u}}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\bar{u} + u}{2}\right) = v$$

It's a "real" vector

$$\overline{w} = \overline{\left(\frac{u - \overline{u}}{2i} \right)}$$

$$= \left(\frac{\overline{u} - \overline{\overline{u}}}{-2i} \right) = \frac{\overline{u} - u}{-2i}$$

$$= \frac{-(\overline{u} - u)}{2i} = \frac{u - \overline{u}}{2i} = w$$

w is also "real". Then, ...

$$v + iw = \frac{u + \overline{u}}{2} + \cancel{i} \, \frac{u - \overline{u}}{2\cancel{i}}$$

$$= \frac{u + \overline{u}}{2} + \frac{u - \overline{u}}{2}$$

$$= \frac{u}{2} + \frac{u}{2} = u$$

$\square$

All we need to write complex vectors like $u = v + iw$ was a conjugation.

Ex $\vec{u} \in \mathbb{C}^3$

$$u = \begin{pmatrix} i \\ 1-i \\ 2+2i \end{pmatrix} = \begin{pmatrix} 0+1i \\ 1-1i \\ 2+2i \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} i$$

$$= \underline{v} + i\,w$$

$$\underline{v} = \frac{u+\bar{u}}{2} = \frac{1}{2}\begin{pmatrix} i \\ 1-i \\ 2+2i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -i \\ 1+i \\ 2-2i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

$$w = \frac{u-\bar{u}}{2i} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Def: Let $L : U \to V$ be a linear map of conjugated complex vector spaces.

Then L is called real if

$$L(\bar{u}) = \overline{L(u)} \quad \forall u \in U. \}$$

Ex: $T : \mathbb{C}^n \longrightarrow \mathbb{C}^m$

in fact $T(z) = Az$ for some complex matrix A.

When is A a real tranformation in sense of the definition above?

$$A\bar{z} = \overline{Az} = \bar{A}\bar{z}$$

$$\Longrightarrow \quad A\bar{z} = \bar{A}\bar{z} \quad \forall z \in \mathbb{C}^n$$

$$\Rightarrow (A - \bar{A})\bar{z} = 0 \quad \forall z \in \mathbb{C}^n$$

$$\overline{(A - \bar{A})\bar{z}} = 0$$

$$(\bar{A} - A)z = 0 \quad \forall z \in \mathbb{C}^n$$

Since $\ker(\bar{A} - A) =$ all of $\mathbb{C}^n$

$$\Rightarrow \bar{A} - A = 0$$

$$A = \bar{A}. \rightsquigarrow (a_{ij} = \overline{a_{ij}})$$

A is a matrix w/ real numbers as entries.

Ex : $\begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$

is a <u>real</u> transformation of complex spaces.

Ex $\frac{d}{dx}: C^1_{\mathbb{C}}[a,b] \longrightarrow C^0_{\mathbb{C}}[a,b]$

is a real transformation

of function vector spaces.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(r(x)+is(x))$$

$$= r'(x) + is'(x)$$

$$\frac{d}{dx}(\overline{f}) = \frac{d}{dx}(r(x) - is(x))$$

$$= r' - is'$$

$$= \overline{\frac{d}{dx}(f)}$$

Non example: $\begin{bmatrix} i & -1 \\ 2 & 1+i \end{bmatrix}$, $i\frac{d^2}{dx^2} + (1-i)\frac{d}{dx}$

not real.

Any transformation

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

OR $$D : C^n_{\mathbb{R}}[a,b] \longrightarrow C^0_{\mathbb{R}}[a,b]$$

becomes a <u>real</u> transformation

of complex vector spaces

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

$$D : C^n_{\mathbb{C}}[a,b] \longrightarrow C^0_{\mathbb{C}}[a,b]$$

<u>Ex</u> : $D = \frac{d^2}{dx^2} + \frac{d^0}{dx^0} : C^2_{\mathbb{C}}[a,b] \to C^0_{\mathbb{C}}[a,b]$

is <u>real</u>. Even if we get complex solutions to D (e^{ix}, e^{-ix})

we can recombine them to real sol'ns by Re, Im.

Thm let $L : U \to V$ be a **real** transformation of complex vector spaces.

Then if $\vec{u}$ is a solution to a linear system $L(u) = 0$.

then so is $\overline{u}$, $\mathrm{Re}(u)$, $\mathrm{Im}(u)$.

(if $u \in \ker L$, then $\overline{u} \in \ker L$
$\mathrm{Re}(u) \in \ker L$
$\mathrm{Im}(u) \in \ker(L)$)

Ex if $e^{ix} \in \ker(D) = \ker\left(\frac{d^2}{dx^2} + \frac{d^0}{dx^0}\right)$

then $\Rightarrow \cos x$, $\sin x$ also.

Pf : Suppose $L(u) = 0$ and L is real.

Then $L(\bar{u}) = \overline{L(u)} = \bar{0} = 0$

L is real

So $\bar{u} \in \ker L$.

Recall $u = v + iw$, where

$v = \mathrm{Re}(u)$

$w = \mathrm{Im}(u)$

$v = \frac{1}{2}u + \frac{1}{2}\bar{u}$. $w = \frac{1}{2i}u - \frac{1}{2i}\bar{u}$.

v, w are linear combinations of $u, \bar{u}$.

Since $\ker(L)$ is a subspace then $v, w \in \ker L$ also. $\square$

Old terminology

adjoint of a matrix

= transpose of the cofactor matrix ✗

Called the adjugate now...

Adjoint is something else...

The adjoint of a transformation is a generalization of the transpose.

(Different from the dual of a transformation)

Def : Let $T : U \to V$ be a transformation of <u>real</u> inner product spaces.

Then the adjoint transformation of T is $\cancel{(T^*)}$ $T^\dagger : V \to U$ (reverses order)

such that

$$\langle T(u), v \rangle = \langle u, T^\dagger(v) \rangle .$$

inner product in V

inner product in U.

Two inner products, one for the domain, one for codomain.

Why does $T^\dagger$ exist? Why is it linear?

Pf: Existence ...

Recall that if V, W are f.d, then all linear functions $W \to \mathbb{R}$ are of the form

$$f(w) = \langle \alpha, w \rangle.$$

$$\langle T(u), v \rangle = \langle u, T^{+}(v) \rangle \qquad (T: U \to V)$$

Note that $\langle T(-), w \rangle : U \to \mathbb{R}$ is a linear map. $(-)$ is the input.

$$\Longrightarrow \quad \langle T(-), w \rangle = \langle -, \alpha_w \rangle$$

Claim is that $T^{+}(w) = \alpha_w$.

Satisfies definition

$$\langle T(u), w \rangle = \langle u, \alpha_w \rangle = \langle u, T^{+}(w) \rangle \checkmark$$

$\langle u, \underline{T^\dagger(v+w)} \rangle$

$= \langle T(u), v+w \rangle$

$= \langle T(u), v \rangle + \langle T(u), w \rangle$

$= \langle u, T^\dagger(v) \rangle + \langle u, T^\dagger(w) \rangle$

$= \langle u, T^\dagger(v) + T^\dagger(w) \rangle \quad \forall u$

$\implies T^\dagger(v+w) = T^\dagger(v) + T^\dagger(w)$

Scalars same. $\square$

<u>Lemma</u>: If $\langle x, y \rangle = \langle x, z \rangle \ \forall x$ $\ast$

then $y = z$. Important!

Pf: $V^\perp = 0$

Since $\langle x, y \rangle = \langle x, z \rangle$

$\langle x, y \rangle - \langle x, z \rangle = 0$

$\langle x, y - z \rangle = 0 \quad \forall x \qquad$ let $x = y - z$

In particular

$\langle y - z, y - z \rangle = 0$

$\implies \|y - z\|^2 = 0 \implies y - z = 0.$

$\square$

Ex: Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. $\mathbb{R}^n, \mathbb{R}^m$ / dot product

(adjoint of A depends on choice of inner product)

What is $A^\dagger$?

- $A^\dagger : \mathbb{R}^m \to \mathbb{R}^n$
- $Au \cdot v = u \cdot A^\dagger v$ $\quad \forall u, v$

$$(Au)^T v = u^T (A^\dagger v)$$

$$u^T A^T v = u^T A^\dagger v$$

$$\Rightarrow \quad u \cdot (A^T v) = u \cdot (A^\dagger v) \quad \forall u$$

$$\Longrightarrow \quad A^T v = A^\dagger v \quad \forall v$$

$A^T = A^\dagger$, so adjoint of a matrix is the transpose w/ dot product.

<u>Ex</u>: $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Recall. all inner products on $\mathbb{R}^n$ are of the form $\langle x,y\rangle = x^T K y$

K positive def. matrix

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$x^T K y = \langle x,y\rangle$

$\langle u,v\rangle = u^T L v$

K is $n \times n$ pos. def

L is $m \times m$ pos. def.

Adjoint of matrix depends on choice of inner product.

What is $A^{\dagger}$ now?

We know that

$$\underbrace{\langle Au, v \rangle}_{\text{on } \mathbb{R}^m} = \underbrace{\langle u, A^+ v \rangle}_{\text{on } \mathbb{R}^n} \quad \forall u, v.$$

$$(Au)^T L v = u^T K A^+ v$$

$$u^T A^T L v = u^T K A^+ v \quad \forall u, v$$

$$\Longrightarrow \quad A^T L = K A^+$$

$$A^+ = K^{-1} A^T L.$$

A^+ is a transpose but changed by K and L.

A symmetric $A^T = A$. $\xrightarrow{\text{general}}$ A is ??* $A^+ = A$

<u>Def</u> : If $L : V \longrightarrow V$

(i.e. a square matrix)

then L is self adjoint if

$L^{\dagger} = L$. (generalization of symmetric)

L is positive def if

$\langle u, T(u) \rangle > 0 \quad \forall u$.

$\left(= \langle T(u), u \rangle \right)$

- $(A^{\dagger})^{\dagger} = A$
- $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$.

Midterm 2 Review:

① All inner products on $\mathbb{R}^n$ are of the form $\langle x, y \rangle = x^T K y$ where K is pos. def.

② All pos. def matrices are symmetric.

$\boxed{\text{Why symmetric?}}$

Remember: Given a matrix A, $q(x) = x^T A x$ is called the quadratic form of A.

$x^T A x$ A is not symmetric *

$$(x \;\; y) \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 2xy + 3y^2$$

$$(x \;\; y) \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 2xy + 3y^2$$

K $\quad K^T = K$

Review:

Two ways of showing that K is positive definite, so far...

- $q(x) = x^T K x > 0 \quad \forall x \neq 0.$
- Show that K is the Gram matrix of some independent vectors

Prop: If K is Gram matrix of $\{v_1 \dots v_k\}$

$$K = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & & \\ \langle v_2, v_1 \rangle & \cdot & & \\ & & \cdot & \\ & & & \ddots \langle v_k, v_k \rangle \end{pmatrix}$$

then $\{v_1 \dots v_k\}$ are independent iff K is positive def.

1. Let $A \in M_{n\times n}(\mathbb{R})$.

(a) $x^T A x = x^T A^T x$

(b) $K = \frac{1}{2}(A + A^T)$ is symmetric

(c) $x^T A x = x^T K x$

(d) If K is pos. def. then

$$(A)_{ii} > 0.$$

<u>Solution:</u>

(a) $x^T A^T x = (Ax)^T x$

$= Ax \cdot x$ by def.

$= x \cdot Ax$ symmetry of dot prod.

$= x^T A x$

(b) $K^T = \left(\frac{1}{2}(A + A^T)\right)^T = \frac{1}{2}(A^T + A^{T^T})$

$= \frac{1}{2}(A^T + A) = K.$

(c) $x^T K x$

$= x^T \left(\frac{1}{2}(A + A^T) \right) x$

$= \frac{1}{2}\left(\underline{x^T A x} + \underline{x^T A^T x} \right)$

$= \frac{1}{2}(2x^T A x) = x^T A x$

(d) If K is pos. def.

then $x^T K x > 0 \quad \forall x$.

let $x = e_i$

$e_i^T K e_i > 0$.

$e_i^T K e_i \underset{(c)}{=} e_i^T A e_i = e_i^T a_i$

a_i = i^{th} column of A

$= (A)_{ii} > 0$

$\square$

4 on review...

Let v and w be independent vectors in $\mathbb{R}^n$. Let $v^\perp$ and $w^\perp$ be the orthogonal complements to $\mathrm{span}(v)$ and $\mathrm{span}(w)$ respectively.

Show that $\dim(v^\perp \cap w^\perp) = n-2$.

<u>Def</u>: Let $W \subseteq V$,

$$W^\perp = \{v \in V \mid \langle w, v\rangle = 0 \ \forall w \in W\}.$$

$$\mathrm{span}(w)^\perp = \{v \in V \mid \langle w, v\rangle = 0\}$$

$$\mathrm{span}(v)^\perp \cap \mathrm{span}(w)^\perp$$

$$= \left\{u \in V \;\middle|\; \begin{array}{l}\langle u, v\rangle = 0 \\ \langle u, w\rangle = 0\end{array}\right\}$$

$$= \mathrm{Span}(v, w)^\perp.$$

How to compute $\operatorname{span}(v,w)^\perp$?

Thm $\ker(A^T) = \operatorname{coker}(A) = \operatorname{img}(A)^\perp$

$\operatorname{coimg}(A) = \operatorname{img}(A) = \ker(A^T)^\perp$

$\dim(\operatorname{img}(A)) = \dim(\operatorname{img}(A^T))$

Let $A = \begin{pmatrix} v & w \end{pmatrix}$ $n\times 2$ matrix.

$\operatorname{span}(v,w) = \operatorname{img}(A)$.

$\begin{pmatrix} v \\ w \end{pmatrix}$ $2\times n$

$v^\perp \cap w^\perp = \operatorname{span}(v,w)^\perp$

$= \operatorname{img}(A)^\perp = \ker(A^T)$.

$\underline{\dim(v^\perp \cap w^\perp)} = \dim(\ker(A^T))$

$= \#$ of columns of A^T

$- \dim($span of columns of $A^T)$

rank nullity

$= \underline{n - 2}$

4.4.10

If $W \subseteq V = \mathbb{R}^n$,

$W = \text{span}(v_1, \dots, v_k)$,

the $\text{proj}_W v = Pv$ where

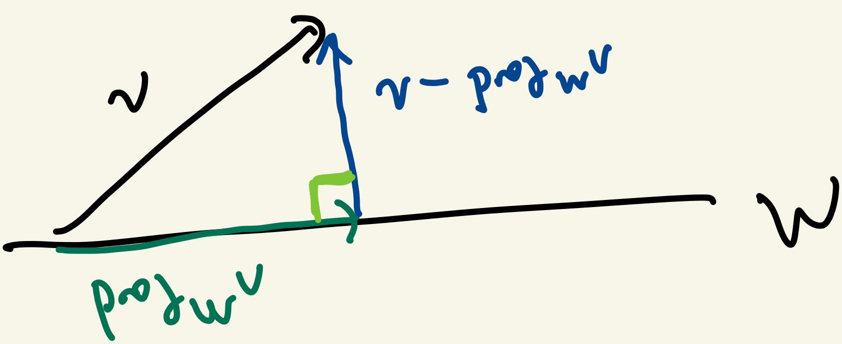
$$P = A(A^T A)^{-1} A^T$$

If $W = \text{span}(u)$ 6.

then $P = I - uu^T$

Remember $\text{proj}_W v$ is the unique vector in W s.t.

$$v - \text{proj}_W v \perp \text{proj}_W v$$



Normally, if W has an orthonormal basis $\{u_1 \ldots u_k\}$

$(|u_i| = 1, \quad u_i \cdot u_j = 0)$

$\left((u_1 \ldots u_k) = Q \text{ orthogonal} \right)$

$$\text{proj}_W v = \langle u_1, v \rangle u_1 + \ldots + \langle u_n, v \rangle u_n$$

$$\in W$$

or P s.t. $Pv = \text{proj}_W v$.

$\exists\ v = v^* + z, \quad v^* = \text{proj}_W v \in W$

$z \in W^\perp$.

unique!

6. Let u be a unit vector.

Let $P = I - \underbrace{uu^T}$

u $n \times \cancel{1}$, u^T $\cancel{1} \times n$

uu^T is $n \times n$

(a) $P^2 = (I - uu^T)(I - uu^T)$

$= I^2 - 2uu^T + uu^Tuu^T$

$= I - 2uu^T + u(u^Tu)u^T$

unit vector $\|u\| = 1$

$u \cdot u = 1$

$u^Tu = 1$

$= I - 2uu^T + uu^T = I - uu^T$

$= P.$

(b) $\operatorname{img}(P)^{\perp}$

$$= \operatorname{coker}(P) = \ker(P^T).$$

$$P^T = (I - uu^T)^T = I^T - u^{T}u^{TT}$$

$$= I - u^T u = P$$

P symmetric

$$\operatorname{img}(P)^{\perp} = \ker(P^T) = \ker(P).$$

Notice that row reduction is not an option.

Then $w \in \ker(P)$

$$\iff Pw = 0$$

$$\iff (I - uu^T)w = 0$$

$$\iff Iw = u\underbrace{(u^T w)}_{\text{scalar}}$$

$1\times n,\ n\times 1$

1×1

$$\iff w = (u\cdot w)u$$

$$\iff w \in \operatorname{span}(u).$$

$$\operatorname{img}(P)^{\perp} = \ker(P)^{\perp} = \operatorname{span}(u).$$