1. Let $A = \begin{pmatrix} 2 & 3 & -2 \\ -3 & 0 & 0 \\ -4 & -6 & 4 \end{pmatrix}$.

a) $\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 2-\lambda & 3 & -2 \\ -3 & -\lambda & 0 \\ -4 & -6 & 4-\lambda \end{pmatrix} = 0$$

$$3 \det \begin{pmatrix} 3 & -2 \\ -6 & 4-\lambda \end{pmatrix} + (-\lambda) \det \begin{pmatrix} 2-\lambda & -2 \\ -4 & 4-\lambda \end{pmatrix} = 0$$

$$3\left(3(4-\lambda) - 12\right) + (-\lambda)\left((2-\lambda)(4-\lambda) - 8\right) = 0$$

$$3(\cancel{12} - 3\lambda - \cancel{12}) + -\lambda(8 - 6\lambda^2 + \lambda^2 - 8) = 0$$

$$-9\lambda + -\lambda(\lambda^2 - 6\lambda) = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$-(\lambda)(\lambda^2 - 6\lambda + 9) = 0$$

$$-\lambda(\lambda - 3)^2 = 0$$

$$\boxed{\lambda = 0}, \quad \boxed{\lambda = 3, 3}$$

(b) $V_0 = \ker(A - 0I) = \ker(A)$

$$= \ker\begin{pmatrix} 2 & 3 & -2 \\ -3 & 0 & 0 \\ -4 & -6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -2 \\ -3 & 0 & 0 \\ -4 & -6 & 4 \end{pmatrix} \xrightarrow{2r_1 + r_3} \begin{pmatrix} 2 & 3 & 2 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{3}{2}r_1 + r_2} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 9/2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{2}{3}r_2 + r_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 9/2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

So $\ker(A) = \begin{pmatrix} 0 \\ 2/3\, z \\ z \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

So the eigenvalue $\lambda = 0$ has alg mult $= 1$ and geom mult $= 1$

$$V_{\lambda=3} = \ker(A - 3I) = \ker\begin{pmatrix} -1 & 3 & -2 \\ -3 & -3 & 0 \\ -4 & -6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 & -2 \\ -3 & -3 & 0 \\ -4 & -6 & 1 \end{pmatrix} \xrightarrow{-r_1\ -r_2\ -r_3} \begin{pmatrix} 1 & -3 & 2 \\ 3 & 3 & 0 \\ 4 & 6 & -1 \end{pmatrix}$$

$$\xrightarrow[-4r_1 + r_3]{-3r_1 + r_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 12 & -6 \\ 0 & 18 & -9 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{6}r_2]{-3/2\, r_2 + r_3} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{3}{2}r_2 + r_1} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}r_2} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

So $V_3 = \ker(A - 3I) = \begin{pmatrix} -1/2\, z \\ 1/2\, z \\ z \end{pmatrix}$

$= \text{span}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

So the alg mult of $\lambda = 3$ is 2 but the geom mult is 1.

(c) We need 1 generalized eigenvector for $\lambda = 3$. Therefore we solve

$$(A - 3I)\, w_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} -1 & 3 & -2 \\ -3 & -3 & 0 \\ -4 & -6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ -3 & -3 & 0 & 1 \\ -4 & -6 & 1 & 2 \end{array}\right)$$

Same steps!

$$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \xrightarrow[-4r_1 + r_3]{-3r_1 + r_2} \begin{pmatrix} 1 \\ -4 \\ -6 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{6}r_2]{-3/2 r_2 + r_3} \begin{pmatrix} 1 \\ -2/3 \\ 0 \end{pmatrix} \xrightarrow{\frac{3}{2}r_2 + r_1} \begin{pmatrix} 0 \\ -2/3 \\ 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}r_2} \begin{pmatrix} 0 \\ -1/3 \\ 0 \end{pmatrix}$$ so

RREF

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ and

$$w_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} z + \begin{pmatrix} 0 \\ -1/3 \\ 0 \end{pmatrix}$$ and we can

use $\begin{pmatrix} 0 \\ -1/3 \\ 0 \end{pmatrix}$ as the generalized eigenvalue.

Therefore the Jordan decomposition is

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 1 & -1/3 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 2 & 1 & -1/3 \\ 3 & 2 & 0 \end{pmatrix}^{-1}$$

2. In matrix form this system is

$$\begin{pmatrix} 2 & 3 \\ -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Therefore the LSS is

$$x^* = (A^T A)^{-1} A^T \vec{b}$$

- $$A^T A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 9 \\ 9 & 11 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{18}\begin{pmatrix} 11 & -9 \\ -9 & 9 \end{pmatrix}$$

- $$A^T \vec{b} = \begin{pmatrix} 2 & -1 & 2 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

So the LSS explicitly is

$$x^* = \frac{1}{18}\begin{pmatrix} 11 & -9 \\ -9 & 9 \end{pmatrix}\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \boxed{\frac{1}{18}\begin{pmatrix} -4 \\ 0 \end{pmatrix}}$$

3.

(a) The vertex-edge formula says that

$$\#v - \#e = 1 - \#\text{ind circ}$$

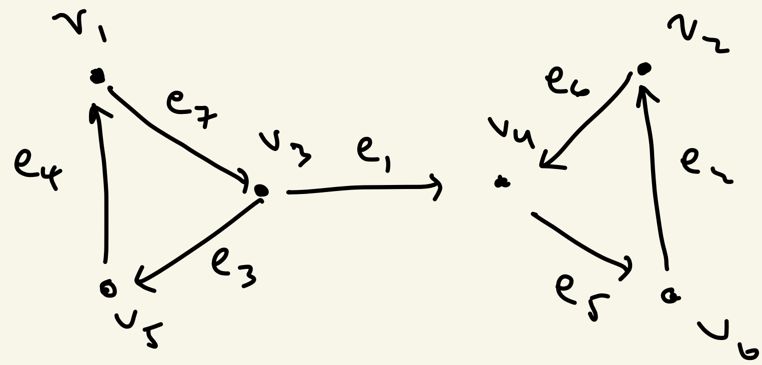
So

$$6 - 7 = 1 - \#\text{ind circ}$$

$$\Longrightarrow \boxed{\#\text{ind circ} = 2}$$

This is expected since the graph has 2 "holes" in it. One given by $e_3 + e_4 + e_7$ and the other $e_2 + e_5 + e_6$.

(b) $\partial(e) = \text{end} - \text{start}$, so we need to label the vertices first.



And a circuit is an element of the kernel of ∂

so

$$\partial(e_1 + e_2 - 2e_3 - 2e_4 + e_5 + e_6 - 2e_7)$$
$$= v_4 - v_3 + \cancel{v_2} - \cancel{v_6} - \cancel{2v_5} + \cancel{2v_3} - \cancel{2v_1} + \cancel{2v_5}$$
$$+ \cancel{v_6} - \cancel{v_4} + \cancel{v_4} - \cancel{v_2} - \cancel{2v_3} + \cancel{2v_1}$$
$$= v_4 - v_3 \neq 0.$$

So this is <u>not</u> a circuit.

4.

(a) Writing $\langle -,- \rangle$ in matrix form we have

$$\langle (x_1, y_1), (x_2, y_2) \rangle$$
$$= (x_1\ y_1)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Bilinearity

- $$\langle \vec{u} + v, w \rangle = (u+v)^T \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w$$
$$= (u^T + v^T)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w$$
$$= u^T\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w + v^T\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w$$
$$= \langle u, w \rangle + \langle v, w \rangle$$

- $$\langle c\vec{v}, \vec{w} \rangle = (cv)^T \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w$$
$$= c\, v^T \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} w = c\langle v, w \rangle$$

✓

The other side is similar.

Symmetry

$$\langle \vec{x}_1, \vec{x}_2 \rangle = 2x_1x_2 - x_1y_2 - x_2y_1 + y_1y_2$$
$$= 2x_2x_1 - x_2y_1 - x_1y_2 + y_2y_1$$
$$= \langle \vec{x}_2, \vec{x}_1 \rangle.$$

✓

Positivity.

Recall that $v^T K v > 0$ for all $v \neq 0$

iff K is positive definite.

But K is positive definite since

$$\det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - 1 = 0$$

$$\lambda^2 - 3\lambda + 2 - 1 = 0$$

$$\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} > 0$$

it's eigenvalues are both positive!

So $\langle v, v \rangle = v^T \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} v > 0$

and $\langle -, - \rangle$ is positive.

(b) $$|2v_1v_2 - v_1w_2 - v_2w_1 + w_1w_2|$$

$$\leq \sqrt{2v_1^2 - 2v_1v_2 + v_2^2}\sqrt{2w_1^2 - 2w_1w_2 + w_2^2}$$

5.

(a) The vectors $\{v_1\ v_2\ v_3\ v_4\}$ do not form a basis of any $\mathbb{R}^n$. These vectors live in $\mathbb{R}^4$ so the only possibility is $n=4$. But they don't form a basis of $\mathbb{R}^4$ either.

They are not <u>independent</u> since $v_3 \in \text{span}(v_1\ v_2)$ according to the RREF. In particular

$$v_3 = -2v_1 + 4v_2 \text{ is a dependency.}$$

They also don't span since $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin$ span of columns of RREF.

Since they neither span nor are independent, they can't be a basis.

(b) $\ker(M) = \ker\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad x = 2z$, $y = -4z$, $w = 0$

$$\ker(M) = \begin{pmatrix} 2z \\ -4z \\ z \\ 0 \end{pmatrix} = \text{span}\begin{pmatrix} 2 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

thus $\dim(\ker(M)) = 1$.

(c). By rank-nullity

$$\dim(\ker(M)) + \text{rank}(M) = 4$$

$$1 + \text{rank}(M) = 4$$

$$\Rightarrow \text{rk}(M) = 3.$$

Alternatively $rk(M) = \#$ of leading 1's

$= 3.$

(d) M is not invertible since

- $rk(M) \neq 4$
- columns don't form a basis
- rows don't form a basis
- RREF $\neq$ Identity

Any of these are good enough answers, by the Fundamental Theorem.

6. First $\exp((x_1,y_1) + (x_2,y_2))$

$$= \exp(x_1+x_2, y_1+y_2)$$

$$= (e^{x_1+x_2}, e^{y_1+y_2})$$

$$= (e^{x_1}e^{x_2}, e^{y_1}e^{y_2})$$

$$= (e^{x_1}, e^{y_1}) +_Q (e^{x_2}, e^{y_2})$$

$$= \exp(x_1, y_1) +_Q \exp(x_2, y_2)$$

Constants also can come out

$$\exp(c(x,y)) = \exp(cx, cy)$$

$$= (e^{cx}, e^{cy})$$

$$= ((e^x)^c, (e^y)^c) = c(e^x, e^y)$$

$$= c(\exp(x,y)).$$

7. (a) $w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\quad w_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ $\quad w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$v_1 = w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1$$

$$= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$v_3 = w_3 - \frac{w_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{w_3 \cdot v_2}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{-1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

Normalizing we get

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(b)

$\text{Proj}_W v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2$ where

u_1, u_2 are an orthonormal basis of W.

From part (a) W has orthonormal basis

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

So

$$\text{proj}_W(2,1,2) = \left(\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \left(\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2}(-1)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}.$$

8.

(a) $A = \begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 6 \\ 3 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} -3 & -3 & 6 \\ 3 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

(b) Since $A^3 = 0$, then

$$e^A = I + A + \frac{1}{2}A^2 + 0 + 0 + \cdots$$

$$= I + A + \frac{1}{2}A^2$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -3 & -3 & 6 \\ 3 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -3 \\ 2 & 3 & -1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -3/2 & -3/2 & 3 \\ 3/2 & 3/2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1/2 & -3/2 & 0 \\ 7/2 & 9/2 & -4 \\ 1 & 1 & -1 \end{pmatrix}}$$

- Alternatively, the longer way is to compute the Jordan form. I had hoped that part (a) was a hint to do it the first way.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & -3 \\ 2 & 2-\lambda & -1 \\ 1 & 1 & -2-\lambda \end{pmatrix} = 0$$

$$-\lambda\big((2-\lambda)(-2-\lambda) + 1\big) + (-3)\big(2 - (2-\lambda)\big) = 0$$

$$-\lambda(-4 + \lambda^2 + 1) + (-3\lambda) = 0$$

$$-\lambda^3 + 3\lambda - 3\lambda = 0$$

$$-\lambda^3 = 0 \implies \lambda = 0, 0, 0$$

$$V_0 = \ker(A) = \ker\begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{Swap } r_1 r_3} \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -1 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{-2r_1 + r_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{+r_2 + r_3} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{2/3\, r_2 + r_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3} r_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $V_0 = \ker(A) = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = \operatorname{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

So Jordan chain is $w_1 = \begin{pmatrix}-1\\1\\0\end{pmatrix}, w_2, w_3,$

$Aw_2 = w_1$ and $Aw_3 = w_2$.

$$\begin{pmatrix}0&0&-3\\2&2&-1\\1&1&-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}-1\\1\\0\end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix}0\\1\\-1\end{pmatrix} \xrightarrow{-2r_1+r_2} \begin{pmatrix}0\\1\\-1\end{pmatrix} \xrightarrow{r_1+r_3} \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\xrightarrow{2/3r_2+r_1} \begin{pmatrix}2/3\\1\\0\end{pmatrix} \xrightarrow{1/3r_2} \begin{pmatrix}2/3\\1/3\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&1&0&2/3\\0&0&1&1/3\\0&0&0&0\end{array}\right) \longrightarrow \begin{array}{l}x = -y + 2/3\\ z = 1/3\end{array}$$

so $w_2 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y+2/3\\y\\1/3\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}y + \begin{pmatrix}2/3\\0\\1/3\end{pmatrix}$

$\Longrightarrow \quad w_2 = \begin{pmatrix}2/3\\0\\1/3\end{pmatrix}$

$$\begin{pmatrix}0&0&-3\\2&2&-1\\1&1&-2\end{pmatrix} w_3 = \begin{pmatrix}2/3\\0\\1/3\end{pmatrix}$$

$$\begin{pmatrix}2/3\\0\\1/3\end{pmatrix} \longrightarrow \begin{pmatrix}1/3\\0\\2/3\end{pmatrix} \longrightarrow \begin{pmatrix}1/3\\-2/3\\2/3\end{pmatrix} \longrightarrow \begin{pmatrix}1/3\\-2/3\\0\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}-1/9\\-2/3\\0\end{pmatrix} \longrightarrow \begin{pmatrix}-1/9\\-2/9\\0\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&1&0&-1/9\\0&0&1&-2/9\\0&0&0&0\end{array}\right) \qquad \begin{array}{l}x = -y - 1/9\\ z = -2/9\end{array}$$

$w_3 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y-1/9\\y\\-2/9\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}y + \begin{pmatrix}-1/9\\0\\-2/9\end{pmatrix}$

So $w_3 = \begin{pmatrix} -1/9 \\ 0 \\ -2/9 \end{pmatrix}$.

Thus

$$\begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2/3 & -1/9 \\ 1 & 0 & 0 \\ 0 & 1/3 & -2/9 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \begin{pmatrix} -1 & 2/3 & -1/9 \\ 1 & 0 & 0 \\ 0 & 1/3 & -2/9 \end{pmatrix}^{-1}$$

So

$$e^{\begin{pmatrix} 0 & 0 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix}} = \begin{pmatrix} -1 & 2/3 & -1/9 \\ 1 & 0 & 0 \\ 0 & 1/3 & -2/9 \end{pmatrix} e^{\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}} \begin{pmatrix} -1 & 2/3 & -1/9 \\ 1 & 0 & 0 \\ 0 & 1/3 & -2/9 \end{pmatrix}^{-1}$$

$$e^{\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}} = I + N + \frac{1}{2} N^2$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad e^{A} = \begin{pmatrix} -1 & 2/3 & -1/9 \\ 1 & 0 & 0 \\ 0 & 1/3 & -2/9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & -1 \\ 3 & 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -3/2 & 0 \\ 7/2 & 9/2 & -4 \\ 1 & 1 & -1 \end{pmatrix}$$

anyway don't do it this way if you can avoid it!

9.

(a). Let $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$.

Then

$$P_A(\Lambda) = c_3 \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}^3 + c_2 \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}^2 + c_1 \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} + c_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= c_3 \begin{pmatrix} \lambda_1^3 & & \\ & \lambda_2^3 & \\ & & \lambda_3^3 \end{pmatrix} + c_2 \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} + c_1 \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} + c_0 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_3\lambda_1^3 + c_2\lambda_1^2 + c_1\lambda_1 + c_0 & & \\ & c_3\lambda_2^3 + c_2\lambda_2^2 + c_1\lambda_2 + c_0 & \\ & & \ddots \end{pmatrix}$$

Since λ_i are roots of the char poly then

$$c_3\lambda_i^3 + c_2\lambda_i^2 + c_1\lambda_i + c_0 = 0$$

Each of the above diagonal entries is 0, so

$$\Longrightarrow \quad P_A(\Lambda) = 0. \qquad \square$$

(b). By the decomposition

$A = S\Lambda S^{-1}$ so

$$P_A(A) = c_3(S\Lambda S^{-1})^3 + c_2(S\Lambda S^{-1})^2 + c_1(S\Lambda S^{-1}) + c_0 I$$

$$= c_3 S\Lambda^3 S^{-1} + c_2 S\Lambda^2 S^{-1} + c_1 S\Lambda S^{-1} + c_0 SIS^{-1}$$

$$= S\left(c_3 \Lambda^3 + c_2 \Lambda^2 + c_1 \Lambda + c_0 I \right) S^{-1}$$

$$= S \, p_A(\Lambda) \, S^{-1} = S \, 0 \, S^{-1} = 0$$

All in all $\quad p_A(A) = 0.$ $\quad\square$