1. Find the permuted LU decomposition of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 4 & -2 \end{pmatrix}.$$

Solution. To row reduce to an upper triangular matrix, first we swap the first row, and then do the row operation $r_1' = 2r_1 + r_2$. Then P corresponds to the permutation which swaps 1 and 3, while L corresponds to the inverse of the above row operation. Then we can conclude that

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & -2 \\ 0 & 7 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. Recall we defined a permutation matrix to be a matrix P such that

$$P \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is a rearrangement of the elements $a_1, \ldots, a_n$. Show that the product of any two permutation matrices is another permutation matrix.

Solution. Given two permutation matrices P and Q, we can show that QP simply rearranges the entries a column vector by P first, then rearranges them again by Q. Doing two rearrangements of a column vector in a row is simply another way to rearrange n elements, so QP is another permutation matrix. In fact if P corresponds to a permutation σ and Q corresponds to a permutation τ, then QP corresponds to the composition $\tau \circ \sigma$.