1. Let $\mathcal{P}$ be the vector space of polynomials in one variable. Let W be the subset of $\mathcal{P}$ consisting of all polynomials $p(x)$ such that $p(5) = 0$. Show that W is a subspace of $\mathcal{P}$.

Solution. First, W is nonempty since $x - 5 \in W$. To show the addition is closed, let $p(x), q(x) \in W$. Then

$$(p+q)(5) = p(5) + q(5) = 0 + 0 = 0.$$

Therefore $p + q \in W$ as well.

Finally, let $c \in \mathbb{R}$. Then $(cp)(5) = c \cdot p(5) = c \cdot 0 = 0$. Thus $cp \in W$, and W is closed under scalar multiplication. In conclusion, W is a subspace.

2. Decide whether the set $S = \{(x, y, z) \mid x^2 + y^2 = z\}$ is a subspace of $\mathbb{R}^3$.

Solution. This is not a subspace. The point $p = (1, 1, 2)$ is in W. But $(-1)p = (-1, -1, -2)$ is not in W. So it is not closed under scalar multiplication, and it is not a subspace.

3. Let V be a vector space and U, W and subspaces. Under conditions is the union $U \cup W$ also a subspace?

Solution. In general, the union of two subspaces is not a subspace. It is most definitely not closed under addition. If $u \in U$ and $w \in W$, then there is no guarantee that $u + w$ is in either U or W. We need a pretty strict condition to make $U \cup W$ a subspace.

Claim: $U \cup W$ is a subspace iff $U \subseteq W$ or $W \subseteq U$.

Assume that $U \cup W$ is a subspace. Assume for contradiction that neither $U \subseteq W$ nor $W \subseteq W$. Then there exists a $u \in U$ such that $u \notin W$ and similarly there exists a $w \in W$ such that $w \notin U$. Then $u + w \in U \cup W$ since it is a subspace. But then either $u + w \in U$ or $u + w \in W$. If the sum is in U, the subtraction shows that $w \in U$ as well which is a contradiction. Similarly if $u + w \in W$, then $u \in W$, which is a contradiction. Thus either $U \subseteq W$ or $W \subseteq U$.

Conversely, assume that $U \subseteq W$ or $W \subseteq U$. In either case $U \cup W = U$ or W, so that it is a subspace. This completes the proof.