1. Let $V = C^0[-1, 1]$ with the L^2 norm. Let $f(x)$ be an odd function and $g(x)$ be an even function. Show that $f(x)$ and $g(x)$ are orthogonal.

Solution. The product of an odd function and an even function is another odd function. So $f(x)g(x)$ is odd. Therefore

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx = 0$$

since the integral of any odd function on a symmetric interval is 0.

2. Let $V = \mathbb{R}^n$ with the dot product. Let A be a matrix and let $v \in \ker A$. Show that v is orthogonal to every row of A.

Solution. By definition, v is such that $Av = 0$. But the ijth entry of Av is the dot product of v with the row a_i. Thus $a_i \cdot v = 0$, and they are orthogonal.

3. Let $V = \mathbb{R}^n$ with the dot product. Let $A = \begin{pmatrix} v_1 \dots v_n \end{pmatrix}$ be an $n \times n$ matrix with columns v_i such that for any $i \neq j$, $\langle v_i, v_j\rangle = 0$ and $||v_i||^2 = 1$. Show that $A^{-1} = A^T$. Such matrices are called orthogonal matrices.

Solution. We show that $A^T A = I$. By definition, the ij entry of $A^T A$ is the dot product of the ith row of A^T and the jth column of A. But the ith row of A^T is the ith column of A. So

$$(A^T A)_{ij} = \langle v_i, v_j\rangle.$$

By assumption, this inner product is 1 when $i = j$ and the 0 when $i \neq j$. Therefore $A^T A = I$ exactly, and $A^T = A^{-1}$.